

Zone of Concept Image Differences in Infinite Limits at Undergraduate Level

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Abstract. *The gap between the concept image and the concept definition will greatly affect a person's understanding of the concept. This study aims to reveal the gap between the concept image and the concept definition, and the causes of the emergence of the gap. The research subjects were 16 first-year mathematics students at one of the universities in Aceh, Indonesia. Data were obtained from the results of written tests and in-depth interviews. This qualitative research with a phenomenological study approach was analyzed descriptively. The results showed that the gap that occurs in the concept includes: the division of a non-zero number by zero yields infinity, the concept of dividing zero by zero is zero, the concept of dividing a number by zero is the same as the concept of the limit, e.g., $\lim_{x \rightarrow 0} \frac{1}{x^n}$, $n \in \mathbb{N}$, and the concept of the left-hand limit as $x \rightarrow 1^-$ means that the satisfied value of x is all negative numbers such that $\lim_{x \rightarrow 1^-} \frac{1}{(x-1)^2} = -\infty$. The emergence of these concept images results from the subject's learning experience at school and a calculus course. This finding provides insights and considerations for educators to comprehensively introduce students to the concept of infinity to avoid misunderstandings in limit and next materials.*

Keywords: *concept image, concept definition, infinite limit, limit function, calculus*

Introduction

Calculus at school and college levels has four key concepts: limits, derivatives, integrals, and the whole set of calculus concepts (Habineza, 2013). The concept of limits is the basis for learning calculus and developing mathematical thinking outside of calculus (Beynon & Zollman, 2015; Ferrini-Mundy & Lauten, 1993; Tall, 1992). The concept of limit has also been widely studied until the present (Juter, 2005; Roh, 2008; Bezuidenhout, 2001; Dubinsky et al., 2005; Sierpńska, 1987) as it is one of the concepts that raises many problems and difficulties among educators and students (Bezuidenhout, 2001; Ferrini-Mundy & Graham 1991; Cornu, 1991). Some of the problems include explaining why the concept of limits is the basis of calculus (Davis & Vinner, 1986) and the formal definition of limit is not used in certain situations (Szydlik, 2000; Williams, 1991; Cottrill et al., 1996), for example, its development in an introductory course in calculus and real analysis (Beynon & Zollman, 2015). In addition, understanding of the limit concept is often incomplete and unstructured (Cornu, 1981).

Misconceptions that appear in the concept of limit are influenced by misconceptions about the concept of infinity. Students who do not understand the infinite process can confuse the concept of limit with the value of a function or sequence or an approximation of a limit (Cottrill et al., 1996). In addition, an infinite process is considered a limit rather than seeing a limit as a result of an infinite process (Vinner, 2002). Such misunderstandings are unavoidable in understanding limits and difficult to correct (Davis & Vinner 1986; Williams, 1991). Thus, misunderstanding limits not only affects the understanding of limits but also causes difficulties in subsequent topics such as continuity, derivatives, and integrals (Sulastrri et al., 2021; Bezuidenhout, 2001; Cornu, 1991; Sierpińska, 1987). This misunderstanding is also related to one's perception of a concept.

Individual perception of an object in interpreting things can be stated as a personal understanding of the concept (Sulastri et al., 2021). How to interpret an object may vary between individuals according to their experience and understanding of the object. There are three phases of perceiving an object: selection by focusing on certain characteristics, organization based on experience, and interpretation based on the selection process and organization (Tall, 1996).

Concepts and conceptions are interpreted similarly to concept image and concept definition (Sfard, 1991; Tall & Vinner, 1981). Conception is considered an internal representation of one's mind against a concept based on experience. Furthermore, a collection of interrelated conceptions will form a concept image. In other words, the concept image constitutes an association, intuition, efficiency, degeneration, and key elements (Przenioslo, 2004). Tall & Vinner (1981) state that concept image is one's knowledge of meaning in the form of mental images, traits, characteristics, and processes associated with the concept. This concept image will be wider if an individual has a rich conception (Attorps, 2006). Thus, it can be stated that the concept image consists of all the cognitive structures in an individual's mind associated with a concept. The cognitive conflict can occur because of the gap or difference between the concept image and the concept definition (Tall & Vinner, 1981). This concept gap can affect a person in understanding and solving a problem related to the concept (Sulastri et al., 2021).

Many studies have been conducted on the concept image in calculus, especially the limit concept. Several studies on the concept image include derivatives (Nurwahyu et al., 2020), anti-derivatives (Moru & Qhobela, 2019), certain integrals (Habineza, 2013), the limit of functions (Sulastri et al., 2021; Oehrtman, 2002; Juter, 2005; Przenioslo, 2004; Alcock & Simpson, 2004, 2005), limit and infinity (Juter & Grevholm, 2006), limit and continuous (Tall & Vinner, 1981), and limit sequence (Roh, 2008). The gap in the concept of limit can occur as the meaning of limit is expressed in inappropriate words, such as "near" or "closer" (Oehrtman 2002; Tall & Vinner 1981; Williams 1991). Thus, to generate a concept image that contains a formal definition of limit, the definition is not only memorized but also applied to build valid arguments (Beynon & Zollman, 2015).

More specifically, Nasr & Haifa (2018) conducted a study on infinite limits involving tenth and eleventh-grade students showed that infinity is conceptualized as an endless number or the largest number. In addition, there are misunderstandings in distinguishing between infinity and innumerable and relating infinity and unbounded. Arnal-Palacián (2020) conducted a study to determine the infinite limit of a sequence by analyzing its phenomenology in a Freudenthal sense. In this case, a definition is chosen and then analyzed using two approaches: an intuitive approach and a formal approach. According to Edwards et al. (2005), the concept of infinite limit can be part of basic or advanced mathematical thinking depending on the related performance. Also, procedural knowledge is more dominant than conceptual knowledge (Sebsibe & Feza, 2019).

In contrast to previous studies, this study focuses on the concept image and the gap in the concept of infinite limit based on phenomenological studies. The participants were students who had studied infinite limits in a calculus course. Therefore, this study aims to reveal students' gap between the concept image and the concept definition of infinite limit, and the contributing factors to the emergence of the concept image.

Method

This study constitutes qualitative research using a phenomenology approach. According to Lester (1999) the phenomenological method is very effective in promoting individual experiences and perceptions from their own perspective. This approach aims to reveal students' gap between the concept image and the concept definition, and the factors that may contribute to the emergence of concept image. To reveal this concept image, the researchers explored students' learning experiences about infinite limits and knowledge relevant to this topic.

The subjects in this study were mathematics students who had studied the limit of functions in the Differential Calculus course during the first semester. There were 16 students from one of the universities in Aceh who participated in the written test. Subsequently, eight students (labeled as S1, S2, ..., S8) were selected for the in-depth interview according to their answers. The criteria

for the chosen answers were the presence of errors or mistakes in solving questions and the diversity of correct and incorrect answers. In addition, the participants were also asked for their consent to participate in the interview.

The written test for the concept of infinite limit discussed in this study consisted of two questions. The first question deals with determining the limits of the given functions, and the second one asks for finding the limit of functions and the meaning of the infinite limit. These questions were adapted from Moru (2006), Jordaan (2005), and Denbel (2014). The didactic design (Suryadi, 2013) and the didactic situation (Brousseau, 2002) were referred to while designing the question on the limit of functions. All the questions were then validated by experts in the field of mathematics to get concepts and problem-solving correctly according to scientific concepts. Subsequently, these test items were tested on high school students of Grade 12 in Bandung and mathematics education students from one of the universities in Bandung, Indonesia. These students were chosen because they have studied the limit of functions in mathematics or Calculus courses. This preliminary test was administered to ensure the readability and accuracy of the questions that would be tested on university students. After several revisions, according to the validation and test results, the questions could be used at the undergraduate level.

Research data comprised written test answers and transcripts of in-depth interview recordings conducted through the Zoom Meeting and Google Meet applications. In phenomenological research, all these data must be analyzed (Lester, 1999) through five stages (Moustakas, 1994) as follows: making a list of expressions from the subject's test answers, reducing and eliminating inappropriate expressions, creating clusters and writing themes for the same and consistent expressions, validating expressions and themes, and making Individual Textural Description (ITD) according to the theme accompanied by the interview excerpts. In this case, data analysis was carried out descriptively to reveal the concept image that occurred and examine the gap between the concepts possessed by the subject based on experience while studying the limit of a function with scientific concepts related to the concept of infinite limit. In addition to that, it examined the factors that may cause the emergence of these gaps.

Results and Discussion

In this study, two problems related to the infinite limit will be discussed. Student answers will be presented by grouping several categories that are considered the same. The presentation of the gap between the concept image and the definition concept, begins with the student's concept of the infinite limit for each question. The first question can be seen in Figure 1.

<p>Determine the value of the following limits:</p> <p>a. $\lim_{x \rightarrow 0} \frac{1}{x} = \dots$</p> <p>b. $\lim_{x \rightarrow 0} \frac{1}{x^2} = \dots$</p> <p>c. $\lim_{x \rightarrow 0} \frac{1}{x^3} = \dots$</p> <p>d. $\lim_{x \rightarrow 0} \frac{1}{x^4} = \dots$</p> <p>e. $\lim_{x \rightarrow 0} \frac{1}{x^{2n-1}} = \dots$ for n Natural numbers</p> <p>f. $\lim_{x \rightarrow 0} \frac{1}{x^{2n}} = \dots$ for n Natural numbers</p> <p>Please draw a conclusion from the limits of the functions above!</p>

Figure 1. First question in the infinite limit

The subjects provided several different answers to the first question. The classification of their answers is depicted in Table 1. Based on the five categories of the answers listed in Table 1, most of the subjects (56%) answered that all the limits of the given functions result in infinity.

This answer was made based on the solution to the function $\frac{1}{x^n}$, which makes the denominator zero due to the substitution of $x=0$. As a result, the function results in division by zero. The subjects' explanations for the written answers are presented below (see Figure 2).

- a. A function divided by zero is infinity.
 - b. A fraction whose numerator remains 1, but the denominator is zero. When divided by zero, the value is infinity.
- For any power of x , when x is 0 and divided by 1, the result is ∞ .

Table 1. Subjects' answer category on the first question

The category of students' answer	Concept understanding
The limits of all functions are the same, that is, infinity	Division by zero produces infinity
The limits of all functions are the same i.e. undefined	Division by zero returns undefined
The limits of all functions are the same, which is zero	Dividing zero by zero yields zero
The limits of all functions are the same, i.e. the limit does not exist	Limit of a function is stated to have a value when the values for the left and right-hand limits are the same
The the limit of the function will be different depending on the odd or even power (n) of the function $\frac{1}{x^n}$	$\lim_{x \rightarrow 0} \frac{1}{x^n}, n \in N$ for n odd numbers, the limit does not exist, while for n even numbers, there is a limit that is infinity

Referring to the subject's explanation about the division by zero, it can be explained that the subject has not correctly comprehended the concepts of limit and infinity. It is indicated by the method used to compute the limit of the function, that is, by directly substituting $x=0$ and ignoring the conditions for a limit to exist. From the first problem, $\lim_{x \rightarrow 0} \frac{1}{x} = \infty$, the subject only understood that division by zero from the function yields infinity, instead of recognizing the limit of the function getting closer to infinity when x gets closer to 0. The interview excerpt with subjects who answered infinity for all limit functions is illustrated below.

- S1 : if there is a limit of the function $\frac{1}{x}$ as x approaches 0, whatever the power it is, the result is infinity. Because when we substitute $x = 0$ into $\frac{1}{x}$, it will get to infinity.
- S5 : any number divided by zero will result in infinity.
- S4 : so we conclude that $\frac{1}{x}$ is infinite and $\frac{1}{x^n}$ is also infinite for the power of n , where n is a natural number.
- S6 : because all return $\frac{1}{0}$, then, the answer is infinite.

In addition to division by zero resulting in infinity, several subjects expressed that division by zero is undefined. This concept is true when viewed from the concept of dividing a non-zero number by zero. However, this case will be different if related to the concept of limit. During the interview, the subject changed his answer from undefined to infinity and reasoned that the result of $\frac{1}{x}$ is infinite, and zero to any power always equals zero. The way he solved the problem shows that the subject misunderstood the concept of limit, especially in determining the limit of a function whose denominator is zero. In this case, the subject perceives the resulting expression as a form of dividing two numbers without paying attention to the concept of limits related to the value of the vicinity. When asked the meaning of x approaching 0, the subject understood that x is not equal to zero but incorrectly applied it.

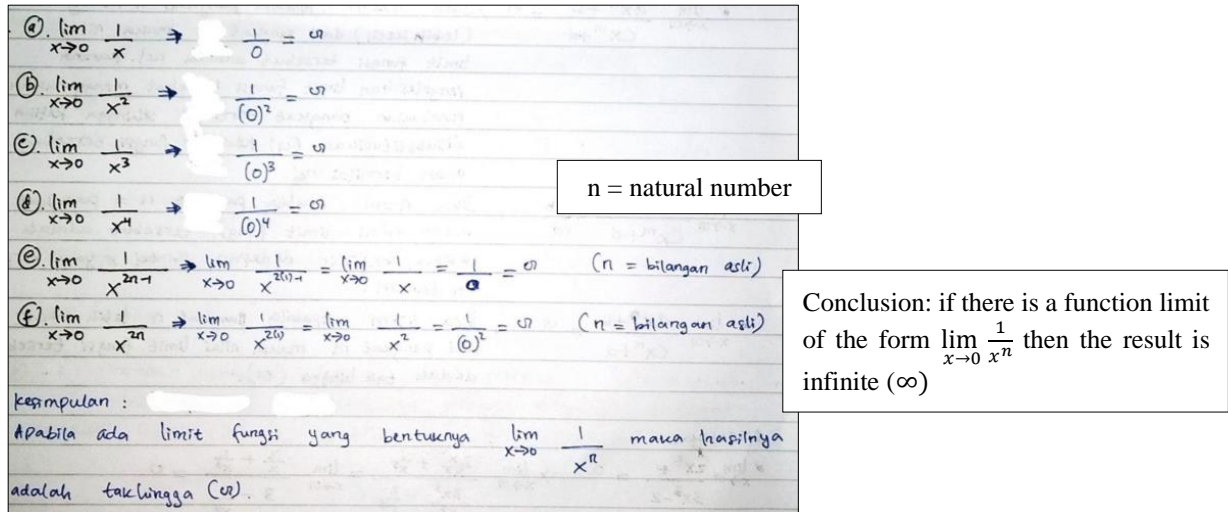


Figure 2. The limits of all functions have the same value, i.e. infinity, by substitution

Furthermore, Subject (S9) stated that the limits for all the given functions are the same, namely zero (See Figure 3). It is obtained by multiplying the numerator and the denominator by least common denominator to avoid the denominator zero after being substituted by $x=0$. In his answer, the subject wrote that "The denominator is 0 and the numerator is a number greater than 0, so I manipulated the initial function such that the limit does not appear undefined". Based on the interview results, he chose this method because the substitution method would produce a zero denominator. According to the subject, a limit has a condition where the function should not have the denominator zero, while the method of multiplying by the conjugate is impossible because the function does not contain a root. He stated, "...I manipulated the function like factoring, but the result is still $\frac{1}{x}$, for example, $\frac{x \cdot 1}{x^2}$ remains $\frac{1}{x}$...". In this case, the subject seems to understand that dividing a non-zero number by the denominator zero will result in undefined. Besides, by manipulating the function, the form of division of zero by zero is obtained. Indirectly, the subject stated that zero divided by zero equals zero, which is an inaccurate concept. After being investigated, it turns out that the subject did not comprehensively understand the concept.

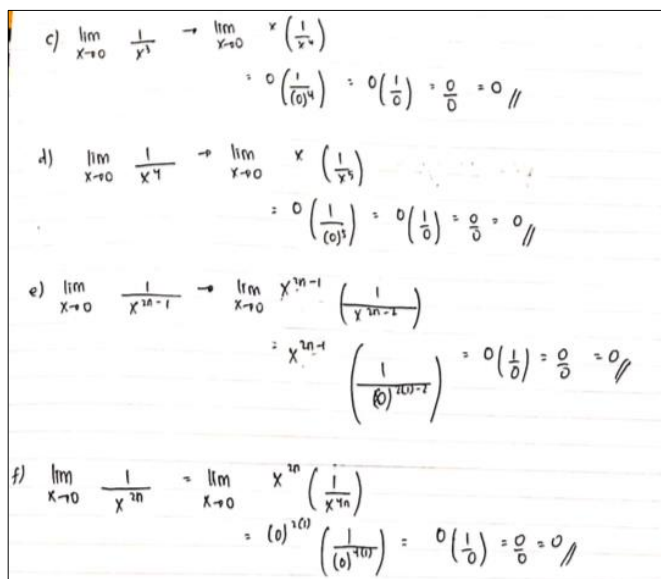


Figure 3. The limits of the functions have the same value, namely zero by manipulating the form of the function

In addition, a subject responded that the limits of all given functions do not exist (see Figure 4). In the solution, for $\lim_{x \rightarrow 0} \frac{1}{x}$, the subject explained the left and right-hand limits. $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$ means that x gets closer to 0 from the left to produce negative infinity, which differs from the result of $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$ as x gets closer to 0 from the right, i.e., positive infinity. Since the right and left-hand limits as x approaches 0 produce different values, the limit of the function $\lim_{x \rightarrow 0} \frac{1}{x}$ does not exist. Such an answer indicates that the subject understands the concept as to what condition a limit is said to exist, namely when the left-hand limit and the right-hand limit have the same value.

From solving $\lim_{x \rightarrow 0} \frac{1}{x}$, the subject understands the concept of limit in which the limit of a function exists when the values of the left-hand limit and the right-hand limit are equal. However, the subject inaccurately responded to other cases, such as $\lim_{x \rightarrow 0} \frac{1}{x^2}$. Solving $\lim_{x \rightarrow 0} \frac{1}{x^2}$ must pay attention to the exponents for the left and right-hand limits. As x gets closer to 0 from the right, the limit will get closer to positive infinity, and as x gets closer to 0 from the left, the limit will also get closer to positive infinity. In other words, the left and right-hand limits have the same value. Thus, $\lim_{x \rightarrow 0} \frac{1}{x^2}$ is infinite.

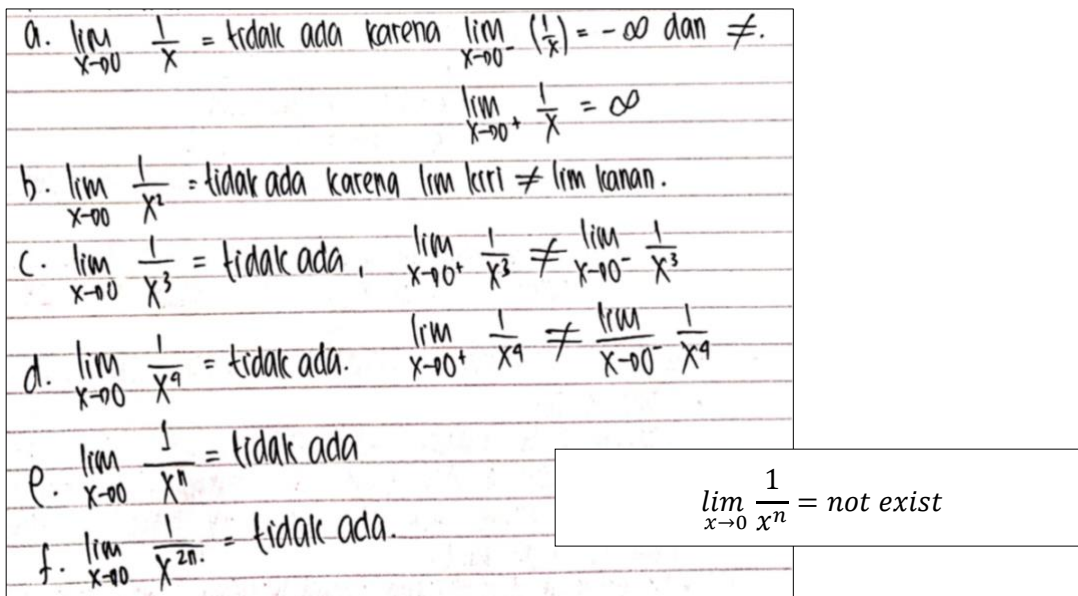


Figure 4. The limits of the functions do not exist because the values for the left and right-hand limits are different

25% of subjects answered correctly for all the limit functions even though at the end they arrived at incorrect conclusions. Only one subject did not make a complete conclusion (see Figure 5) by stating that " $\lim_{x \rightarrow a} \frac{b}{x^n}$ will have a value if n is an even number and $b \neq 0$ ". This answer indirectly shows no limit exists when n is odd, as described in the following interview excerpt.

S2 : ... $2n-1$ is a negative number. For odd numbers, when $x \rightarrow 0$, it makes negative. The negative will remain negative if the power is odd. No matter the value, if it is odd, it still does not have a limit because the left-hand limit must be negative while the right-hand limit must be positive, and this case is specifically for x approaching 0...

The interview results confirm that the subject grasped limits of the function concept, especially for $\lim_{x \rightarrow 0} \frac{1}{x^n}$ related to rules for odd and even exponents n. For the limit of a function with an odd power (n), the subject stated that the left-hand limit must be negative while the right-hand limit is positive. In this case, the subject only mentions that for any values of x, as x approaches 0 from the left, which means x is a negative number, the value is still negative. This signifies that when x is a negative number, then f(x) is also a negative number. This explanation is inaccurate because the context is the limit of a function, not merely a function, even though the subject understands the concept of limit by stating that "the limit is close to but not at that value". In other words, the subject less precisely concluded that when x approaches zero from the right, the limit approaches infinity. On the other hand, the limit of the function as x approaches zero from the left is negative infinity.

Meanwhile, for the limit of a function with an even power (n) as in $\lim_{x \rightarrow 0} \frac{1}{x^{2n}}$, the subject pointed out that $\lim_{x \rightarrow 0} \frac{1}{x^{2n}}$ will yield infinity from the left and right-hand sides. As x approaches 0 from the left, where x is a negative number, the limit of the function will be a positive number if the function is squared. This value will be equal to the limit value as x approaches 0 from the right, where x is a positive number. The subject explained that infinity is obtained by substituting x=0 into the function $\frac{1}{x^{2n}}$. Therefore, the subject understands that $\frac{1}{0}$ is infinity.

Handwritten notes in Indonesian:

- a. $\lim_{x \rightarrow 0} \frac{1}{x} = \text{tidak ada, karena nilai limit kiri} \neq \text{limit kanan}$
- b. $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$, karena $\frac{1}{0} = \text{tak hingga}$
- c. $\lim_{x \rightarrow 0} \frac{1}{x^3} = \text{tidak ada}$
- d. $\lim_{x \rightarrow 0} \frac{1}{x^4} = \infty$
- e. $\lim_{x \rightarrow 0} \frac{1}{x^{2n-1}} = \text{tidak ada}$
- f. $\lim_{x \rightarrow 0} \frac{1}{x^{2n}} = \infty$

kesimpulannya, untuk nilai n bentuk $\lim_{x \rightarrow a} \frac{b}{x^n}$ akan memiliki nilai jika n bernilai adalah bilangan genap dan $b \neq 0$.

Printed conclusion box:

None, because limits of the functions from the left and the right are not the same

Conclusion:
For the form $\lim_{x \rightarrow a} \frac{b}{x^n}$ will have value if n is an even number and $b \neq 0$

Figure 5. The limit of the function exists when n is even

On the other hand, another subject articulated that "it (undefined/none) applies to odd powers because limits of the functions from the left and the right are not the same". In this case, the subject notices that the limit of a function exists if the limits from the left and the right have the same value. Although the answer is correct for every limit of the given functions, the subject's conclusion for the entire limits of the functions is inaccurate, as mentioned below:

These functions have no limit when x goes to zero because the solution will return $\frac{1}{0}$ and the value of $\frac{1}{0}$ is undefined/none.

It applies to odd powers because the left and right-hand limits are not the same.

For even powers, the result is always $+\infty$ or $-\infty$ because the left and right-hand limits go to infinity as x approaches 0.

The statement that "the value of $\frac{1}{0}$ is undefined/none" confirms that the subject misunderstood the concepts of division by zero, limit, and infinity. The quotient of $\frac{1}{0}$ is equal to infinity. For odd powers, the subject mentioned that no limit exists because the limit values from

the left and right-hand sides are different. In this case, the subject does not clearly and completely elaborate the difference.

If it is associated with $\lim_{x \rightarrow 0} \frac{1}{x^n}$ for n even numbers, the limit of the function will arrive at infinity from the left and the right. Thus, the limit of the function is infinity. For an even power, the subject stated that it yields $+\infty$ or $-\infty$ and interpreted that the positive infinity and negative infinity are equal, which is a wrong concept. $+\infty$ and $-\infty$ are two symbols that have different meanings. In the concept of limit, if it produces a positive infinite from the left and a negative infinite from the right, the limit values from the two sides are different. Hence, the limit of the function does not exist. Based on the interview results, we may conclude that the subject does not thoroughly comprehend the context of the question. Several concepts are not derived from their own thinking but rather from looking at other sources while answering the question. The subject also admitted that he was confused in distinguishing between infinite and undefined. As such, the subject stated that $\frac{1}{0}$ results in undefined or none.

One subject accurately answered the limits of all given functions, which are generally in the form of $\lim_{x \rightarrow 0} \frac{1}{x^n}$ for n even or odd. In the written answer, the subject did not give any justifications for each answer to the limit functions. The conclusion given is also inaccurate because he explains positive and negative exponents. In fact, in the context of the given questions, there is nothing to do with positive and negative numbers. As such, the subject's answer is considered wrong in writing even and odd powers. Whereas by his explanation, he means that the limit of a function turns to infinity when the function has an even power, and there is no limit when the function has an odd power (see Figure 6).

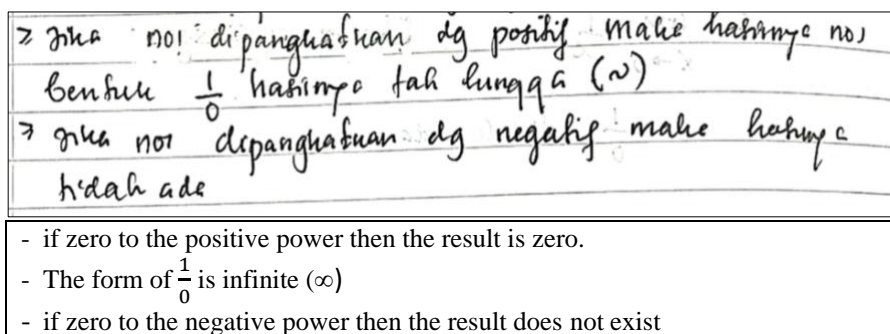


Figure 6. the answer to all limit functions is correct, but the conclusion is incorrect.

Another subject answered correctly, accompanied by quite detailed solution steps (See Figure 7) despite no explanation given for each step. From the solutions, it is evident why $\lim_{x \rightarrow 0} \frac{1}{x^n}$ obtains positive infinite, negative infinite, and no limit.

In the case of odd powers, for example, given the limit of a function with n to the power of 1 or $\lim_{x \rightarrow 0} \frac{1}{x}$, then for the right-hand limit ($\lim_{x \rightarrow 0^+} \frac{1}{x}$), the closer x approaches 0 from the right, where x is a positive number, the limit will approach positive infinity. While for the left-hand limit ($\lim_{x \rightarrow 0^-} \frac{1}{x}$), as x gets closer to 0 from the left, where x is a negative number, the limit is getting closer to negative infinity. Thus, since the left and right-hand limits are different, the limit of the function does not exist. This condition applies to all odd numbers because n is any negative number substituted into a function with odd powers that will also produce a negative number.

In the case of even powers, for example, given $\lim_{x \rightarrow 0} \frac{1}{x^2}$, if the value of x is a positive number (x is getting closer to 0 from the right), the limit will yield a positive number. It means that as x approaches zero from the right, the limit gets closer to positive infinity. On the other hand, for x a negative number (x is getting closer to 0 from the left), then if the value of x is substituted into the function, the limit will produce a positive number due to the process of squaring x. Thus, as

x approaches zero from the left, the limit value will approach positive infinity. The results indicate that the left-hand limit and the right-hand limit possess the same value, positive infinity. Therefore, the limit of the function exists.

a. $\lim_{x \rightarrow 0} \frac{1}{x} = \text{tidak ada}$

Jwb:

$\lim_{x \rightarrow 0^+} \frac{1}{x} = \frac{1}{0^+} = +\infty$

$\lim_{x \rightarrow 0^-} \frac{1}{x} = \frac{1}{0^-} = -\infty$

karena $\lim_{x \rightarrow 0^+} \neq \lim_{x \rightarrow 0^-}$
Maka $\lim_{x \rightarrow 0} = \text{tidak ada}$

Because the left and right-hand limits are different, so the limit of the function does not exist

b. $\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$

Jwb:

$\lim_{x \rightarrow 0^+} \frac{1}{x^2} = \frac{1}{(0^+)^2} = +\infty$

$\lim_{x \rightarrow 0^-} \frac{1}{x^2} = \frac{1}{(0^-)^2} = +\infty$ km limit kanan dan kiri sama yaitu $+\infty$

Because the left and right-hand limits are same, $+\infty$

Figure 7. The limit is infinite when n is even, and no limit exists when n is odd

The second question discussed in this study can be seen in Figure 8. The subjects provided various answers to this question. The classification of their answers is listed in Table 2.

Compute the following limit: $\lim_{x \rightarrow 1^+} \frac{1}{(x-1)^2}$ and $\lim_{x \rightarrow 1^-} \frac{1}{(x-1)^2}$.

Does the limit of the function exist? Draw and Explain!

Figure 8. Second question in the infinite limit

Table 2. The category of subjects' answers for the second question

The category of students' answer	Concept understanding
The limit of this function is $\frac{1}{10^{-2}}$	The right-hand limit value is $\frac{1}{(0.1)^2}$, the left-hand limit value is $\frac{1}{(-0.1)^2}$, then, $\frac{1}{(0.1)^2}$ will equal $\frac{1}{(-0.1)^2}$ because the square of negative results in positive. When $\frac{1}{(0.1)^2}$ is negated, the element becomes $\frac{1}{10^{-2}}$.
The limit of this function is infinity	The limit of a function exists if the right-hand limit is the same as the left-hand limit. The left and right-hand limits are infinite. Based on the existence of the limit function theorem, we get $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = \infty$
The limit of the function is zero	Dividing zero by zero equals zero
The right-hand limit is infinity, and the left-hand limit is negative infinity	The left-hand limit as $x \rightarrow 1^-$ means that the value of x is all negative numbers such that $\lim_{x \rightarrow 1^-} \frac{1}{(x-1)^2} = -\infty$
No limit exists	The fraction function has no value or is undefined if the denominator is 0

Most of the subjects correctly determined the left and right-hand limits of the function in their written answers. However, when explored further through interviews, their understanding of the concept was not correct yet. The subject understands that a limit exists as the values of the left and right-hand limits are the same.

Several different solutions are given to determine the left and right-hand limits of $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$. They include using tables to calculate the values of x and f(x), factoring $(x - 1)^2 = (x - 1)(x + 1)$, the direct substitution of x=1, expanding the quadratic form $(x - 1)^2 = x^2 - 2x + 1$, and multiplying by a known factor.

The strategy of using a table is depicted in Figure 9. In this strategy, the subject takes the value of x very close to 1 from the left-hand or the right-hand sides. In the table to determine $\lim_{x \rightarrow 1^-} \frac{1}{(x-1)^2}$, the subject did not solve the square of the negative number; the result of f(x) is thereby unclear. Furthermore, the subject concluded that the left and right-hand limits are equal to the value of $\frac{1}{10^{-2}}$. However, this result is questionable, as no process is previously stated to describe the result. Besides, the subject depicted the graph correctly even though it was not neat. The interview excerpt with the subject is described as follows.

- SI : It can be concluded that the limit value from the right as $x= 1.09$ is $\frac{1}{(0.1)^2}$, while the limit as x approaches 1 from the left is $\frac{1}{(-0.1)^2}$. So, $\frac{1}{(0.1)^2}$ will equal $\frac{1}{(-0.1)^2}$ because the square of negative is positive. Thus, $\lim_{x \rightarrow 1^+} \frac{1}{(x-1)^2}$ is equal to $\lim_{x \rightarrow 1^-} \frac{1}{(x-1)^2}$, that is, $\frac{1}{10^{-2}}$. So, $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$ has a value as shown in the graph.
- P : Does it mean the limit exists? What is your justification?
- SI : Yes, it is. Because it is close to 0.1 from the right. From the left, it is close to 0.1 as well.
- P : How to read it from this graph?
- SI : When $\lim_{x \rightarrow 1^+} \frac{1}{(x-1)^2}$ goes to infinity
- P : What is the limit for this function?
- SI : Infinite
- P : There is nothing written here about infinity
- SI : It is only shown from the graph

In this case, the subject understands that the right-hand limit or the left-hand limit as x approaches one for the function f(x) will head towards infinity, which contrasts his written answer $\frac{1}{10^{-2}}$. This finding reveals that the subject has sufficient ability and understanding of the limit concept. However, the subject does not understand the concept of infinity. The subject expressed that infinity is an infinite number and part of a real number. The graph depicted by the subject is also inaccurate because several points are misplaced, such as when $x=2$, then y should be 1.

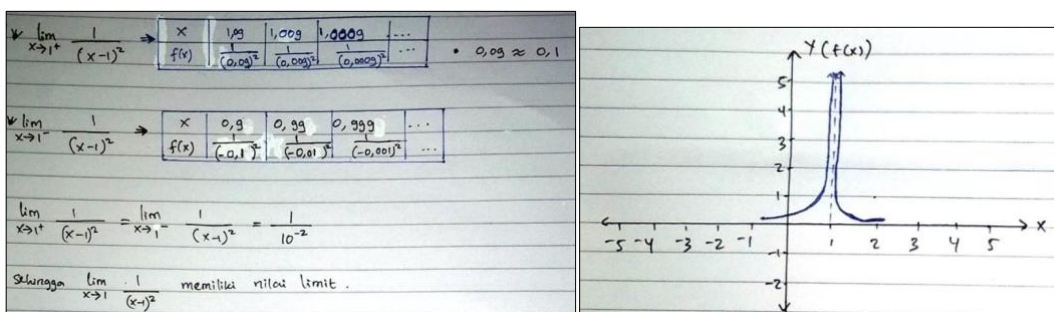


Figure 9. The answer to the limit is $\frac{1}{10^{-2}}$ and the graph of the limit function

Moreover, a subject (S8) determined the limit value by expanding the quadratic equation $(x - 1)^2 = x^2 - 2x + 1$ (see Figure 10). The subject's answer informs that he does not understand what it means by x approaches 1 from the left and x approaches 1 from the right. The subject immediately substitutes the value of $x=1$ for the left and right-hand limits. In addition to the conceptual error, the subject also experiences procedural errors when computing $(1)^2 - 2(1) + 1$, which equals 2. From the graph, the subject demonstrates that the limit of $f(x)$ when x approaches 1 from the left and right sides is one. In the graph, there is a closed circle as a sign that the limit of the function is defined at $\frac{1}{2}$ or $F(1) = \frac{1}{2}$. Otherwise, an open circle denotes that the limit is undefined at $x=1$. At this point, the subject misunderstood the concepts of limit, continuity, and the meaning of defined at a point. He also did not understand how to correctly draw the graph although the subject suggested to determine some values of the point x before illustrating the graph. After being confirmed through interviews, the subject inaccurately calculated $(1)^2 - 2(1) + 1$.

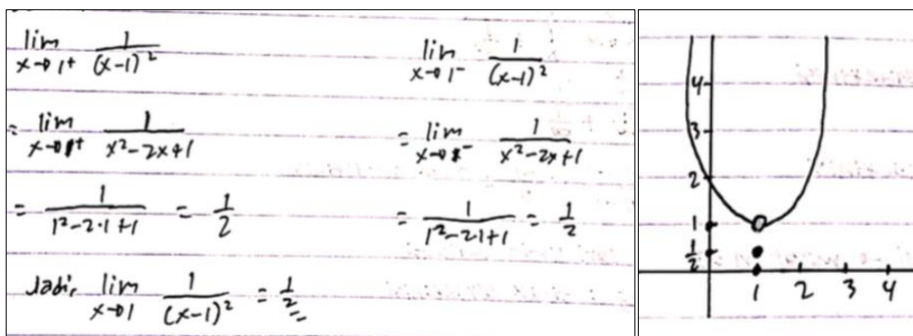


Figure 10. Solving problems and graphs with conceptual and procedural errors

Further, a subject solved the limit of the rational function in an unusual way, multiplying the numerator and the denominator by least common denominator (see Figure 11). According to the subject, this method aims to avoid the denominator zero because dividing a non-zero number by zero will result in undefined. The subject understands the concept of dividing a non-zero number by zero, which is undefined. However, he incorrectly found the division of zero by zero, which resulted in zero. Even though the subject initially avoids division by zero, the result remains the same. Thus, there is an erroneous concept of division by zero and multiplication by the common factor as the given function. Because the concept and solution are incorrect, the graph depicted is also incorrect. From the graph, the subject stated that the limit exists for $\lim_{x \rightarrow 1^+} \frac{1}{(x-1)^2}$ and $\lim_{x \rightarrow 1^-} \frac{1}{(x-1)^2}$. The subject also stated that the limit is defined at zero from the right and left sides. The subject admitted that he incorrectly sketched the graph; thus, he corrected the graph. Again, the subject misrepresented the new graph, especially when determining appropriate points or x values for the graph. In other words, the subject's ability to draw a graph is relatively low.

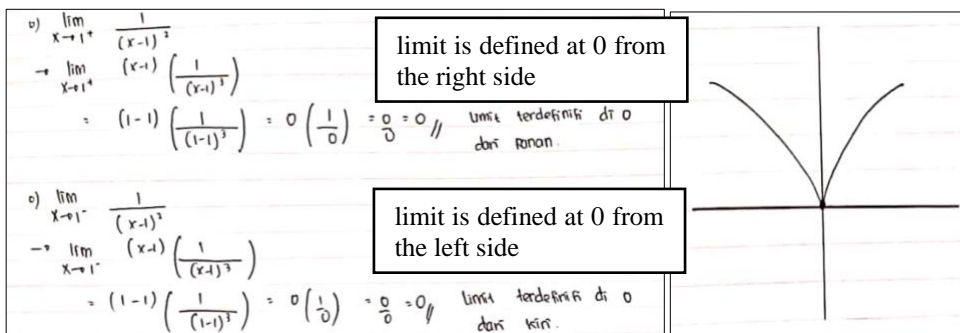


Figure 11. Solving problems with conceptual errors in dividing zero by zero

Most subjects answered correctly for the left and right-hand limits, namely infinity, although the problem solving process varies without a complete explanation. One of the subjects (S1) solved the problem by substituting $x=1$ into the function by counting the position of the value taken, namely x to the right (positive numbers) or x to the left (negative numbers) from zero (see Figure 12). Based on the written answer, an error is found when looking at x getting closer to 0, whereas the question asks for the limit of the function as x approaches 1 from the left and the right. The interviews clarify that the subject understood the concept of limit well, namely, the limit means close to, rather than right at a point. According to the subject, the limits of the given functions exist, i.e., positive infinity from the left and the right because the functions contain even powers. As such, as x is approached from anywhere, the result is positive infinity. The subject understands that the context of this problem is an infinite limit. According to the subject, the graph of a function with infinite limits will continue to go up or down and get closer to a point.

Handwritten work showing the calculation of limits for the function $f(x) = \frac{1}{(x-1)^2}$ as x approaches 1 from both sides. The student writes:

 a) $\lim_{x \rightarrow 1^+} \frac{1}{(x-1)^2} = \lim_{x \rightarrow 1^+} \frac{1}{(0^+)^2} = +\infty$

 b) $\lim_{x \rightarrow 1^-} \frac{1}{(x-1)^2} = \lim_{x \rightarrow 1^-} \frac{1}{(0^-)^2} = +\infty$

 Below these, the student concludes: $\lim_{x \rightarrow 1^+} = \lim_{x \rightarrow 1^-} = +\infty$ artinya nilai bilangan dan $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = +\infty$ atau ∞ .

Figure 12. Answers with the same left and right-hand limit values, namely infinity

Similarly, other subjects also randomly take the value of x , for example, taking $x=-1$ for the value of x from the left of 1. Then, the substitution to the limit of the function will result in infinity. On the other hand, the subjects take $x=1$ for the value of x from the right of 1; thereby substitution result is also found to be infinite. From this written answer (S5 and S7), it is unclear whether the subjects comprehensively understand the concept of limit, especially the left and right-hand limits (see Figure 13). Although taking the same value of x and producing the same limit, the way they describe the graph is different. S5 understands that the value of x to determine the limit is $x=1$, thereby resulting in the correct graph.

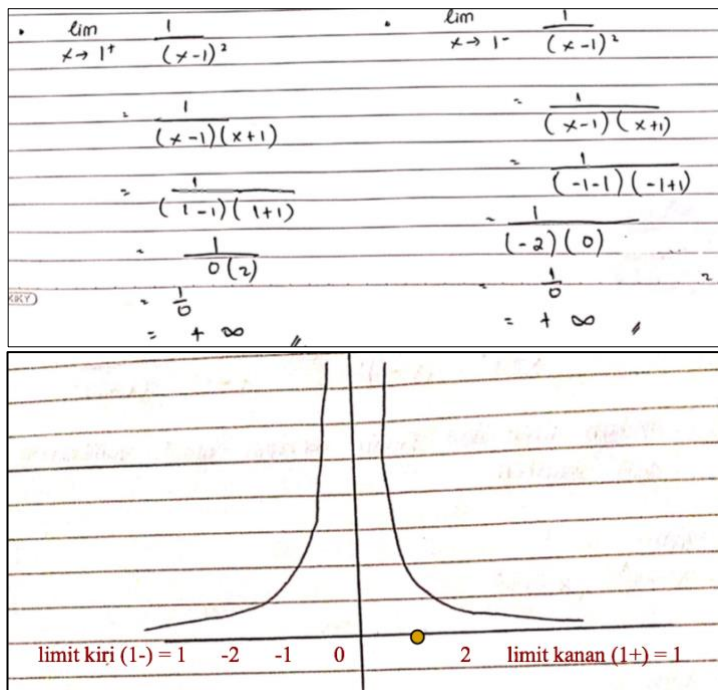


Figure 13. Finding the limit by substituting $x=1$ for the right-hand limit and $x = -1$ for the left-hand limit

In contrast, S7 carelessly determines the value of x for the limit. He also does not put any numbers for the x -axis and the y -axis. The interview with S7 confirms that the graph of the function from the left and the right is bounded by the y -axis, namely when $x=0$. The limit of the function is close to 1, obtained from the substitution of $x=2$ (for the right-hand limit) and $x=0$ (for the left-hand limit). At this point, the subject does not understand what numbers satisfy the rule of "x approaches 1 from the left and the right." $x=0$ is not eligible as it is an asymptote. The subject's explanation during the interview is incorrect and contradicts his previous answer to the left and right-hand limits.

The subjects who gave the correct answer—the left and right-hand limits were the same, i.e. infinity—drew the graph correctly (see Figure 14). The subjects' description related to the graph of the limit function is provided below:

- a. The graph of two limit functions will continue to be positive and go to infinity.
- b. As x gets closer to 1, the limit of the function gets closer to infinity
- c. As x approaches 1 from the left and the right, the limit value gets closer to infinity

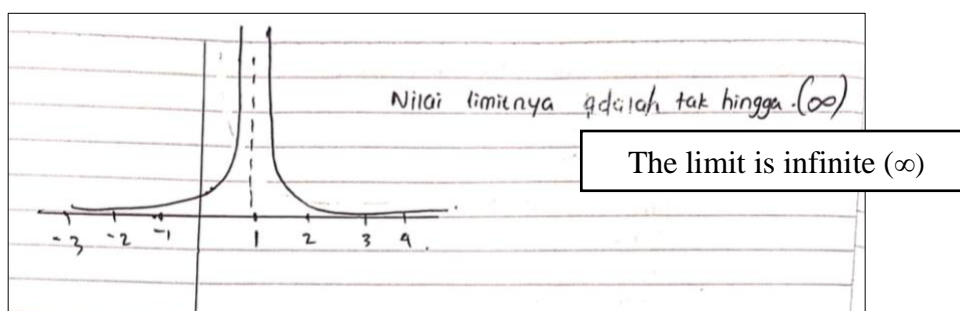


Figure 14. the correct graph for $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$

One concept image given by the subject includes division by zero, which results in infinity (e.g. $\frac{m}{0}, m \neq 0$). This concept is true according to Euler (1770), who states that a non-zero number divided by zero gives infinity (∞). The results of his research also reveal that multiplying by 0 gives more than one solution. Nowadays, this concept is considered incorrect. It can be contradicted by the statement that if a number divided by zero gives infinity and infinity multiplied by zero can produce any number, then all numbers are the same. According to Cajori (1929), this statement is incoherent. However, an exception applies to zero as a divisor (Ohm, 1828). Based on this case, several well-known mathematicians stated that the division of a number by zero is infinite. This problem has found a solution with a more precise concept in limit theory (Paolilli, 2017).

The subject's perception of division by zero resulting in infinity is derived from the previous learning process at school and college levels in basic algebra or calculus courses. In the learning process, instructors do not explain in detail the concepts of infinity, undefined, and indeterminate. Even though it is mentioned in the curriculum, the material presented does not specifically explain this concept. However, the instructors can insert details of this concept more thoroughly in the lesson since the misconception of this concept greatly affects students' understanding of the subsequent material.

Thus, the division of a number n (where $n \neq 0$) by zero results in undefined or is considered impossible. This concept applies only to division and is totally different when it applies to limit theory. This basic concept should be a great concern for educators. Thus, it is important to provide relevant cases or examples that students easily understand. For instance, concerning the inverse concept, $\frac{1}{0}$ equals undefined simply because there is no 0^{-1} . This indicates that 0 is undefined in a multiplication inverse (Neely, no year).

Concept gaps also occur in determining the value of x that satisfies the limit of the function from the left. As $x \rightarrow 1^-$, the subject takes x negative for the left-hand limit, yielding negative

infinity ($\lim_{x \rightarrow 1^-} \frac{1}{(x-1)^2} = -\infty$). In this case, the subject only takes x negative numbers, whereas there are also many positive numbers as x approaches 1 from the left. Besides, the subject incorrectly performed arithmetic operations on the denominator. Even though x is a negative number, the square of the negative number remains positive. Thus, the left-hand limit of the function will be equal to the right-hand limit.

The misconceptions at infinite limits are likely caused by the delivery of material, presentation of material in textbooks, and the order of material in the curriculum. The presentation of the infinity concept and the difference between the concepts of infinity, undefined and indeterminate are incomplete. As a result, some students were confused about when to use infinity, undefined, and indeterminate. Several calculus books, such as Varberg et al. (2007), discuss infinite notation on the topic of inequalities and absolute values. However, the concept of infinity is not clearly and completely discussed. The use of the symbols ∞ and $-\infty$ is only introduced in interval notation; for example, $(3, \infty)$ represents all real numbers greater than 3. An error occurs when the symbol ∞ is often expressed as the largest number and part of a real number. Actually, it is only a symbol to denote the largest number, not included in the set of numbers.

However, in the limit concept, especially limits at infinity and infinity, writing $x \rightarrow \infty$ at limits at infinity is used as a shorthand way of saying that x gets bigger and bigger indefinitely. In addition, for an infinite limit, for example, $\lim_{x \rightarrow a^+} f(x) = \infty$ can be interpreted that when x is close to a from the right, the function will expand infinitely. Conversely, $\lim_{x \rightarrow a^-} f(x) = -\infty$ means that when x is close to a from the left, the function will shrink infinitely. Furthermore, in determining whether a limit exists or not, the conditions must be satisfied in accordance with the limit theorem (Varberg et al., 2007), as follows.

$$\lim_{x \rightarrow c} f(x) = L \text{ if and only if } \lim_{x \rightarrow c^-} f(x) = L \text{ and } \lim_{x \rightarrow c^+} f(x) = L$$

In general, school mathematics does not provide an opportunity for students to develop ideas or concepts about infinity (Eisenmann, 2002). Whereas, learning infinity can lead to the development of cognitive abilities (Jirotková & Littler, 2003). This is relevant to the study conducted by Brackett (1991) in exploring students' thinking and cognitive development related to infinity limits. The contribution of his study includes lesson plans on the concept of limits relevant to the curriculum and grade level. In addition, by understanding students' concept images, educators can implement learning aligned with student needs (Siagian et al., 2021).

Conclusion

Students' concept image in infinite limit is incredibly diverse. Several subjects understand the concept correctly in accordance with the scientific concept. However, their understanding of the concept is not comprehensive. Some of the subjects correctly comprehend the concept because they have memorized or encountered the same problem. In addition, some subjects have perceptions that are not in accordance with the scientific concept. The findings highlight several conceptual gaps between the concept image and the concept definition of infinite limits, as follows: the concept that the quotient of any non-zero number divided by zero is infinite; the concept of dividing zero by zero is zero; the concept of dividing a number by zero is the same as the concept of limit, for example, $\lim_{x \rightarrow 0} \frac{1}{x^n}$, $n \in \mathbb{N}$; and the concept of the left-hand limit as $x \rightarrow 1^-$, it means that the value of x that satisfies the function is all negative numbers so that $\lim_{x \rightarrow 1^-} \frac{1}{(x-1)^2} = -\infty$. The emergence of these concept images and gaps results from the subject's learning experience at school and a calculus course. In the learning process, the concept of infinity is not comprehensively explained, resulting in an incomplete understanding of the infinity concept.

For further research, the researcher will use a didactic design for limits of functions, especially infinite limits. The design contains a thorough study of the concept of infinity and

infinite limits. The goal is that students can easily understand and apply the concepts appropriately. These findings can serve as considerations for educators in comprehensively introducing the concept of infinity to avoid misunderstandings in limits and subsequent materials.

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