

The Thinking Process of Students in Proving Ceva's Theorem in Basic Geometry Course

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Abstract. *The thinking process is crucial in identifying students' challenges in formulating logical arguments, particularly when proving geometric concepts such as Ceva's Theorem. This study investigates the cognitive processes of students with high and medium mathematical ability to solve proof problems related to Ceva's Theorem. Adopting a qualitative approach, the research employed a case study design, utilising data collection techniques including tests, interviews, observations, documentation, and triangulation. The research subjects comprised two undergraduate students of Mathematics Education, representing a medium and high mathematical ability category. The research findings revealed that students with high mathematical ability progressed through assimilation, accommodation, and equilibrium stages in proving Ceva's Theorem. In contrast, students with moderate mathematical ability experienced the disequilibrium phase first before advancing to assimilation, accommodation, and equilibrium. The fundamental difference between these two groups lies in their ability to identify the question posed in the problem and correctly identify the theorem to be proved. This research provides valuable insights into the cognitive processes involved in mathematical proof, offering an in-depth understanding of how students approach the proof of Ceva's Theorem.*

Keywords: *de Ceva's theorem, proof, thinking process*

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Introduction

Proficiency in preparing mathematical proofs is one of the essential competencies students need as an integral part of the learning process in mathematics (Netti et al., 2016; Noto et al., 2019). The development of mathematics, supported by a series of theorems, makes mastery of techniques and methods of proof a must for students (Akmaliyah, 2013). NCTM (1993) stated that mathematical proof systematically applies definitions, postulates, and proven statements through deductive reasoning, creating logical arguments to verify statements. Proofs are essential, fostering students' logical, analytical, and structured thinking skills (Firmasari & Sulaiman, 2019). For anyone wishing to delve into mathematics, a deep understanding of how to construct proofs or, at the very least, the ability to understand the structure and logic behind them is a must. (Wu, 1996). When a student is not competent in constructing mathematical proofs, he/she is likely to face significant difficulties in the overall mathematics learning process (Netti et al., 2016).

Evidence indicates that many students continue to struggle with problems related to mathematical proof. Reflina (2020) found that these challenges arise from several factors, including low language literacy, the perception that mathematical problems are inherently complex, and a lack of interest and aptitude in the field. In addition, these obstacles also stem from students' limited understanding of mathematical concepts, the use of language and mathematical symbols, the selection of appropriate proof strategies, and the ability to read and understand existing proofs (Mujib, 2019). Ko and Knuth (2009) observed that none of the 36 students involved in their study succeeded in constructing complete and thorough mathematical proofs. Moreover, only about 25% of the participants could correctly provide a counterexample within the given problem's context.

Mastering the basics of triangles is essential; a lack of understanding significantly affects angles, space, and trigonometry comprehension (Santiago & Alves, 2022). This statement aligns with Ubah and Bansilal (2019), who asserted that a deep understanding of the basic concepts of triangles is crucial in strengthening the understanding of more complex mathematical structures. Students who can apply their knowledge through Higher Order Thinking (HOT) processes are considered to have superior competence in understanding concepts in depth compared to those who simply rely on rote methods (Zimmerman et al., 2016). Proving Ceva's Theorem enhances students' understanding, offering a deeper grasp of triangle-related concepts and mathematical reasoning skills (Rodriguez, 2016). Ceva's theorem is essential in geometry, proving triangle concurrency and effectively analysing angle bisectors under specific conditions. For instance, Oxman and Sigler (2020) showed how Ceva's theorem facilitates understanding of complex concurrency relationships in triangles, offering foundational proofs that support more advanced studies in geometry. For students in the early stages of learning geometry, grasping the concept of line concurrency presents challenges, as it requires understanding abstract relationships among geometric elements. This difficulty is compounded by the need to visualise and manipulate intersections in various geometric configurations, such as triangles where concurrent lines play a significant role in proofs, including those involving Ceva's Theorem. Research has shown that the abstract nature of concurrent lines, often combined with spatial reasoning skills, can impede comprehension when not supported by visual aids and interactive learning tools (Sunzuma, 2023). Students often struggle with systematic proofs, highlighting the need for deeper research into their cognitive processes in problem-solving.

On the other hand, mathematical proof involves the thinking process. As described by Abdussakir (2014), the thinking process involved in the preparation of evidence can provide deeper insights into identifying the various difficulties students face as they attempt to solve mathematical problems. Likewise, Perbowo and Pradipta (2017) stated that proof requires

thinking. Mathematical proof includes applying a deductive thinking process, in which the understanding and application of logical principles are used systematically to reach valid conclusions based on predetermined premises (Masfingatin & Maharani, 2019). Thus, the thinking process functions as a mental activity carried out by students to construct proofs to solve mathematical problems.

The thinking process involves acquiring, processing, storing, and recalling information to support problem-solving and decision-making effectively (Rahmat, 2019). It is a way out of the problem (Sari & Saragih, 2019). According to Ormrod (2008), the process by which individuals respond to new events using existing schemas is known as assimilation. In contrast, accommodation refers to the mechanism by which a new stimulus or event is understood through modifying existing schemas, or by developing new schemas to accommodate the information more appropriately. According to Piaget (1969), the process of assimilation and accommodation will continue until an equilibrium occurs in solving a problem. In assimilation, students' cognition aligns with incoming information, understanding mathematical problems without altering their schemas (Nizlel et al., 2016; Subanji, 2011). Stimulus integrates into existing schemas; accommodation modifies or creates schemas, aligning cognition with problem complexity and demands (Subanji & Supratman, 2015; Sukoriyanto et al., 2016). The process of assimilation and accommodation can occur simultaneously in problem-solving efforts that involve theorem proving. In such situations, students need to integrate new information with existing knowledge while adjusting to their cognitive schemas to effectively address the complexity of the problem (Piaget, 1969). Thinking is a complex cognitive process involving disequilibrium, assimilation, accommodation, and equilibrium for decision-making and problem-solving.

Several studies have examined the thinking process, such as research by Yohanie et al. (2016), which analysed the thinking process of second-semester students with high, medium, and low learning achievement in solving proof problems. This study showed that each student's process in solving mathematical proofs varies due to their unique thinking and analysis styles (Yohanie et al., 2016). Similarly, Sari and Saragih (2019) analysed three students with visual, auditorial, and kinesthetic learning styles. Their findings indicated that the thinking processes of students with high, medium, and low ability levels followed Polya's stages with relatively similar patterns in relation to academic achievement. However, the thinking processes of students who utilise their learning styles in the context of direct proof were superior to those relying on indirect proof, particularly in systematically and logically understanding problems and selecting appropriate problem-solving strategies.

Rahmah (2021) explored the geometric thinking profile of students in solving geometry problems in terms of Van Hiele's level of thinking. This study found level 0 subjects made errors

in all geometric cognitive aspects. Level 1 subjects succeeded in visualisation and construction but struggled with reasoning due to calculation errors, while level 2 subjects completed visualisation, construction, and reasoning accurately. While previous research mainly focused on van Hiele's thinking stages and Polya's problem-solving strategy, gaps remain in exploring thinking processes via Piaget's frameworks. Proof-based problems can clarify thinking mechanisms, underscoring the need to study students' thinking patterns and dynamics when solving Ceva's theorem proofs, especially among those with high and medium mathematical abilities.

Method

This qualitative research employed a case study approach to explore students' thinking stages in solving proofs of Ceva's Theorem, providing detailed insights and comprehensive findings (Aberdeen, 2013). The research consisted of three stages: preparation, implementation, and conclusion. The preparation stage included selecting the research location, creating validated Ceva's Theorem proof questions, and developing interview guidelines. The implementation stage involved revising instruments, further validating, collecting data, and performing a comprehensive analysis.

The research subjects were two undergraduate students enrolled in the Mathematics Education program at the University of Jember. The research location was selected because no similar studies had been conducted at the institution, and the respondents had previously studied Ceva's Theorem. This study focused on analysing the thought processes of students with differing levels of cognitive ability, namely high (S1) and medium (S2), in solving proof problems related to Ceva's Theorem. Subjects were selected using random sampling techniques to minimise potential bias and ensure representative results. Student abilities were classified based on their final grades in geometry courses, divided into two main categories: high ability and medium ability, as detailed in Table 1.

Table 1. The classification of students based on academic ability levels

No.	Student Capability	Criteria
1	High	$x \geq 81$
2	Medium	$71 \leq x \leq 80$

The criteria for categorizing students into high and medium ability is presented in Table 1, where x is the average score of the mathematics ability. The main instruments were a test and interview guidelines. A long-answer problem on Ceva's theorem was given to 29 students to explore thinking patterns. The data collection technique involved a combination of tests and interviews. Interviews were conducted to explore students' thinking processes when working on the problems. The interview was unstructured, allowing the researcher to adjust and expand

questions based on the situation and dynamics that emerged during the process. The stages of data analysis included data condensation, data presentation, and conclusion drawing. Data validity was guaranteed through the application of method triangulation.

Prove that if point P is arbitrary of ΔABC and points D, E, F , lie on the sides $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ respectively, then If the three-line segments, AD , BE and CF are concurrent, the following holds:

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

Figure 1. Test instrument

To examine the students' thinking process, the research was guided by Piaget's (1969) framework regarding *disequilibrium*, *assimilation*, *accommodation*, and *equilibrium*. This research adopted Piaget's framework modified by Susanto (2011), as presented in Table 2.

Table 2. Indicators of thinking processes according to Piaget

Thinking Process	Indicators
<i>Disequilibrium</i>	1) Students seemed silent for a long time and did not provide answers when asked about the known and requested information in the problem. 2) Students show confusion in determining the relevant triangle height. 3) Students had difficulty when drawing the triangular shape as requested in the question. 4) Students still feel confused when asked to identify congruent triangles. 5) Students seemed to have difficulty in understanding and solving the problems given. 6) Students could not mention the theorem that should be used in solving the problem. 7) Students show confusion when assessing or ensuring the correctness of their work. 8) Students are not able to provide a re-explanation of the steps they have taken to obtain the final result.
<i>Assimilation</i>	1) Students gave spontaneous, correct and incorrect answers related to the known and requested information in the problem. 2) Students can identify and determine the height of triangles relevant to the problem. 3) Students draw the triangle shape as requested in the problem, although the result can be right or wrong. 4) When asked to name congruent triangles, students responded spontaneously, with correct and incorrect answers. 5) Students successfully solve the problem as requested, showing an understanding of the solution procedure. 6) Students spontaneously answer both right and wrong when mentioning the theorem used in solving the problem. 7) Students perform calculations in solving problems correctly
<i>Accommodation</i>	1) Students seemed to be silent momentarily before finally giving the correct answer regarding the known and requested information in the problem. 2) Students were able to draw a triangular shape, although initially incorrect, but consistently made efforts until finally producing the correct drawing. 3) Students managed to mention the congruent triangles, although through several attempts and corrections from the initially wrong answer to reach the right answer.

Thinking Process	Indicators
	4) Students try to determine the solution to the given problem, even if it is initially wrong, and continue to try until they finally find the correct answer. 5) Students were silent for a while before revealing the final results of their work. 6) Students carefully re-examine their work and the underlying reasoning, even though it takes a long time to reach the correct result.
Equilibrium	1) Students can provide appropriate and logical explanations to support the answers that have been submitted. 2) After going through a series of assimilation and accommodation processes, students drew a triangular shape per the request in the problem. 3) Through the application of the assimilation and accommodation process, students are able to identify and state congruent triangles. 4) The answers that students believe correct are obtained through a series of mental processes involving continuous assimilation and accommodation. 5) Students have explained in detail each step in obtaining results through systematic calculations, ensuring that all steps taken are accurate and correct. 6) Students deliver the final results of their work appropriately and as expected.

Results and Discussion

The analysis relies on written data and interviews, uncovering students' thinking stages in proof-solving using Piaget's components: disequilibrium, assimilation, accommodation, and equilibrium.

Interpreting

Figure 2 presents high-ability students' (S1) answers. Based on the answer sheet collected, it can be seen that the subject did not include information related to what was known and requested in the problem. Nevertheless, the subject seemed able to solve the problem correctly, so that the theorem referred to in the problem was successfully proven correctly.

Handwritten mathematical proof for the area of a triangle with a line segment from a vertex to the opposite side. The proof uses area formulas and algebraic manipulation to show that the ratio of the segments of the base is equal to the ratio of the areas of the two smaller triangles.

Left side of the page:

$$\frac{L_{\Delta APF}}{L_{\Delta BPF}} = \frac{\frac{1}{2} \cdot AF \cdot t}{\frac{1}{2} \cdot FB \cdot t} = \frac{AF}{FB} \Rightarrow L_{APF} = \frac{AF}{FB} \cdot L_{BPF}$$

$$\frac{L_{\Delta BPD}}{L_{\Delta DPC}} = \frac{\frac{1}{2} \cdot BD \cdot h}{\frac{1}{2} \cdot DC \cdot h} = \frac{BD}{DC} \Rightarrow L_{BPD} = \frac{BD}{DC} \cdot L_{DPC}$$

$$\frac{L_{\Delta CPE}}{L_{\Delta EPA}} = \frac{\frac{1}{2} \cdot CE \cdot x}{\frac{1}{2} \cdot EA \cdot x} = \frac{CE}{EA} \Rightarrow L_{CPE} = \frac{CE}{EA} \cdot L_{EPA}$$

Bottom left diagram: Triangle ABC with point P on BC. Line segment AP is drawn.

Right side of the page:

$$L_{\Delta BPC} - L_{\Delta CPE} = L_{\Delta BEC} - L_{\Delta CPE}$$

$$\frac{CE}{EA} (L_{\Delta BAE} - L_{\Delta EPA}) = L_{\Delta BEC} - L_{\Delta CPE}$$

$$\frac{CE}{EA} = \frac{L_{\Delta BEC} - L_{\Delta CPE}}{L_{\Delta BAE} - L_{\Delta EPA}}$$

$$\frac{CE}{EA} = \frac{L_{\Delta CBP}}{L_{\Delta APB}}$$

Bottom right diagram: Triangle ABC with point P on BC. Line segment AP is drawn.

Figure 2. The answer of high ability (S1) subject

The following transcript presents the in-depth interview results to further explore the thinking behind the subject's answer.

- R : Have you ever done a problem like this before?*
S1 : Yes, I have
R : When was it?
S1 : It was taught in high school.
R : Tell me what is known about the problem.
S1 : There is a point P in $\triangle ABC$. Points D, E, and F lie on sides BC, AC and AB, respectively. If the line segments AD, BE and CF intersect at P. Prove that $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.
R : Okay, what is being asked in the question?
S1 : (Pause) Prove that $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.

The interview revealed that the subject correctly identified the theorem, knowns, and requests, demonstrating an assimilation process followed by equilibrium in their thinking. This is evident as the subject could answer questions related to the theorem being proven and the information provided in the problem by applying prior knowledge. This finding aligns with Swastika et al. (2023), who observed that while unpacking the source, subjects underwent assimilation, characterised by identifying known information from the problem. Similarly, it corroborates Nizlel et al. (2016), who found that in the early stages of unpacking the source part of the metacognitive process students recognised test problems as familiar, prompting them to recall previously learned material, thus indicating assimilation. Additionally, the subject demonstrated accommodation by pausing before articulating the question, explaining initial steps, and reasoning through the problem-solving process, as evidenced in the interview transcript.

- R : What was the first step you took to solve the problem?*
S1 : First, find the congruent triangles.
R : Which triangles are congruent?
S1 : $\triangle APF$ and $\triangle BPF$, $\triangle BPD$ and $\triangle CPD$, $\triangle CPE$ and $\triangle APE$.
R : Show me the height of the triangle!
S1 : $\triangle PAF$ and $\triangle BPF$ have the same height, which is PF. The height of $\triangle BPD$ and $\triangle CPD$ is PD. As for $\triangle CPE$ and $\triangle APE$, the height is PE.

The subject demonstrated assimilation and achieved equilibrium by accurately identifying congruent triangles, providing justifications, and recognising elements such as the height of a triangle. This finding aligns with Swastika et al. (2023) study, which highlighted that during the early stages of coordination activity, subjects undergo an assimilation process characterised by their efforts to connect mastered concepts to represent various elements in a graph. Furthermore, the subject articulated their response to the question, as evidenced in the interview transcript. The

subject exhibited a systematic thinking process, offering logical justifications for their answers, aligning with the equilibrium indicators in cognitive processing stages.

R : Explain the steps you used to solve the problem!

S1 : $\frac{L\Delta APF}{L\Delta BPF} = \frac{AF}{FB}$ so $L\Delta APF = \frac{AF}{FB} \cdot L\Delta BPF$.

$\frac{L\Delta BPD}{L\Delta DPC} = \frac{BD}{DC}$ so $L\Delta BPD = \frac{BD}{DC} \cdot L\Delta DPC$.

$\frac{L\Delta CPE}{L\Delta EPA} = \frac{CE}{EA}$ so $L\Delta CPE = \frac{CE}{EA} \cdot L\Delta EPA$.

$L\Delta APC$ is the same as $L\Delta ACF - L\Delta APF$, that's already known $L\Delta ACF = \frac{AF}{FB} \cdot L\Delta BCF$ then for $L\Delta APF = \frac{AF}{FB} \cdot L\Delta BPF$.

Then distribute $\frac{AF}{FB} \cdot (L\Delta BCF - L\Delta BPF) = L\Delta ACF - L\Delta APF$. Next, both segments

are equally divided by $L\Delta BCF - L\Delta BPF$ so that we get $\frac{AF}{FB} = \frac{L\Delta CAP}{L\Delta CBP}$. The method is

the same $L\Delta BPC = L\Delta BEC - L\Delta CPE$

$L\Delta BEC = \frac{CE}{EA} \cdot L\Delta BAE$ while $L\Delta CPE = \frac{CE}{EA} \cdot L\Delta EPA$, we obtained

$\frac{CE}{EA} \cdot (L\Delta BAE - L\Delta EPA) = L\Delta BEC - L\Delta CPE$, so we got

$\frac{CE}{EA} = \frac{L\Delta BEC - L\Delta CPE}{L\Delta BAE - L\Delta EPA}$ and $\frac{CE}{EA} = \frac{L\Delta CBP}{L\Delta APB}$.

R : Can you explain the steps for the next one?

S1 : The next one is similar to the previous one $L\Delta APB = L\Delta BAD - L\Delta BPD$. It is

known that $L\Delta BAD = \frac{BD}{DC} \cdot L\Delta CAD$, so $L\Delta BPD = \frac{BD}{DC} \cdot L\Delta CPD$. We got

$\frac{BD}{DC} \cdot (L\Delta CAD - L\Delta CPD) = L\Delta BAD - L\Delta BPD$. So that we get $\frac{BD}{DC} = \frac{L\Delta APB}{L\Delta CAP}$. Then, we

obtained $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{L\Delta CAP}{L\Delta CBP} \cdot \frac{L\Delta CBP}{L\Delta APB} \cdot \frac{L\Delta APB}{L\Delta CAP}$. Next, we crossed out $L\Delta CAP$ with

$L\Delta CAP$ then $L\Delta CBP$ with $L\Delta CBP$ and $L\Delta APB$ with $L\Delta APB$. Finally, the result is equal to 1.

Explaining

The subject showed no confusion, identifying information and specifying requirements fluently, indicating assimilation followed by equilibrium when naming and recalling the theorem. When the subject mentioned what was asked in the problem, he paused momentarily before precisely expressing the question in question, indicating that the subject was going through an accommodation process. This finding aligns with the study of Alghar (2022), who revealed that at the action stage, students show the ability to comprehensively understand the problem, identify related information, formulate what is known and what is asked, and even find hidden information in the problem, especially mathematical information contained in the figure. This finding also aligns with previous studies (Ajai & Imoko, 2015; Indrawati & Tasni, 2016), which found that women tend to be meticulous, skilful, and careful when faced with mathematical problems. The subject's identification of congruent triangles and their heights indicated assimilation with equilibrium. His systematic approach and justifications for answers showed clear equilibrium,

fulfilling the cognitive process indicators. This was characterised by the subject's statement: "Then we obtained $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{L\Delta CAP}{L\Delta CBP} \cdot \frac{L\Delta CBP}{L\Delta APB} \cdot \frac{L\Delta APB}{L\Delta CAP}$, we crossed out $L\Delta CAP$ with $L\Delta CAP$ then $L\Delta CBP$ with $L\Delta CBP$ then and $L\Delta APB$ with $L\Delta APB$, finally the result is 1." Figure 3 presents the subject's thinking process.

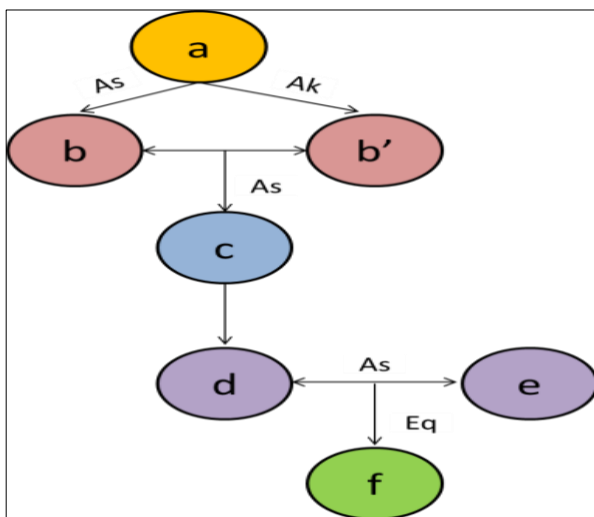


Figure 3. The thinking process of high ability subject (S1)

Description:

a : Test question to prove Ceva's theorem
b : Known in the problem
b' : Asked in the problem
c : Theorem to be proved
d : State the height of a triangle
e : List the congruent triangles

f : Conclusion
 As : Assimilation
 Ak : Accommodation
 Eq : Equilibrium
 Dq : Disequilibrium

Interpreting

Figure 4 presents the answer sheet of medium ability subject (S2).

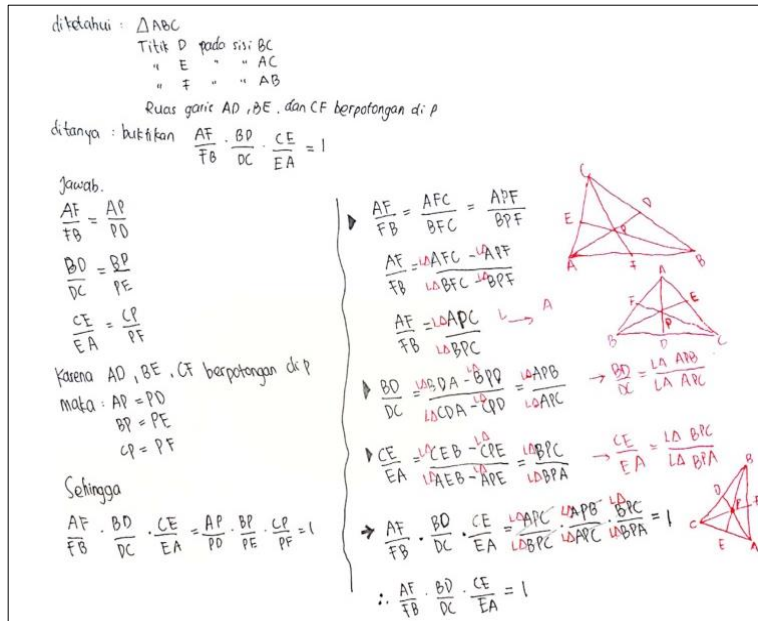


Figure 4. The answer of medium ability subject (S2)

Figure 4 shows that the subject accurately detailed known information and the problem, correctly proving the core theorem. An interview transcript further explores the subject's responses and reasoning.

- R : Have you ever done a similar problem before?
- S2 : It was just a comparison.
- R : When was it?
- S2 : In the past, during a geometry course, we were assigned to prove theorems in front of the class.
- R : Was it in the first semester?
- S2 : Yes.
- R : What is known from the question?
- S2 : So, it is known from the question that there is a $\triangle ABC$, then point D is on the BC side, point E is on the AC side and point F is on the AB side. So, from there the line segments AD, BE and CF intersect at P.
- R : Now, tell me what is asked in the question.
- S2 : We are asked to prove that $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$
- R : Okay, $\frac{BD}{DC}$ is that true?
- S2 : $\frac{BD}{DC}$
- R : Okay, it can be repeated!
- S2 : So, we are asked to prove $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$

Subject S2 demonstrated assimilation accompanied by disequilibrium, recognising the theorem but initially misstating it. Indicators of accommodation emerged as S2 self-corrected, accurately identifying the known information from the problem. This aligns with the findings of Swastika et al. (2023), which stated that during the activity of unpacking the source, the subject

undergoes an assimilation process characterised by identifying known elements of the problem. S2 appeared to experience an assimilation process with a disequilibrium phase, evident when he successfully identified the question but initially missed the relevant line segment.

Subsequently, S2 exhibited the accommodation process, as demonstrated by his efforts to correct himself and accurately restate the question. This is consistent with Subanji (2021), who explained that adaptation involves assimilation and accommodation. Based on established indicators, the subject was considered to have undergone accommodation when he restated the theorem to be proven correctly. Following this, S2 was asked about the initial steps in solving the problem and the reasoning underlying his approach, as illustrated in the accompanying interview transcript.

- R : *In the answer sheet, you wrote APF and $\frac{AF}{FB}$ it is the same as $\frac{APF}{BPF}$. What is that comparison?*
- S2 : *E...What is the comparison? I am confused.*
- R : *Okay, let's see $\frac{AF}{FB}$, what it is in $\triangle APF$ and $\triangle BPF$?*
- S2 : *$\frac{AF}{FB}$, it is as a base.*
- R : *Well... what is the triangle formula related to the base?*
- S2 : *Is it $\frac{1}{2} \times \text{base} \times \text{height}$?*
- R : *What is the formula called?*
- S2 : *The formula for the area of a triangle.*
- R : *Well... about the first step you used to solve the problem, what kind of comparison it was?*
- S2 : *Comparison of areas of triangles.*

Subject S2 initially entered a state of disequilibrium, struggling to justify the statement on the answer sheet. However, with the help of guided questions, S2 reached accommodation by identifying the comparison of triangle areas as the first step in solving the problem. This aligns with Sukoriyanto et al. (2016), who observed that subjects with intermediate abilities often engage in the accommodation process to understand a problem, as evidenced by their challenges. For instance, this is reflected in their study when a subject faced difficulty relating the number of groups to the number of members in each group but persisted in finding the correct solution, demonstrating the accommodation process.

According to established indicators, S2 demonstrated assimilation when identifying congruent triangles and providing justifications. This finding corresponds to Swastika et al. (2023), who stated that subjects engage in assimilation during the initial coordination activity, characterised by their ability to connect known concepts to represent elements on a graph. This process is illustrated in the following interview transcript.

- R : *Do you know what is the height of the triangle in the problem?*

- S2 : *F for that $\frac{APF}{BPF}$, the height is PF. for $\frac{APB}{APC}$, AP is the height. And for the last one, $\frac{BPC}{BPA}$ the height is BP.*
- R : *Can you see the relationship between one figure and another?*
- S2 : *It can be congruent.*
- R : *Okay, can you mention which shapes you are referring to?*
- S2 : *So, the congruent ones are $\frac{APC}{BPC}$, $\frac{APB}{APC}$ and $\frac{BPC}{BPA}$.*
- R : *Do you mean that APC is congruent with BPC, APB is congruent with APC and BPC is congruent with BPA?*
- S2 : *Yes*

After disequilibrium, assimilation, and accommodation, subject S2 reached equilibrium, explaining problem-solving steps smoothly and accurately. This finding aligns with Subanji (2011), stating that when student learn, they will experience disequilibration. With the condition of disequilibration, there will be a process of assimilation and accommodation until equilibrium. S2 reached the equilibrium process when he successfully mentioned the final stage in solving the proof of Ceva's Theorem, by basing his answer on the equations he had previously described. This process can be seen in the following interview transcript.

- R : *Explain the steps you used to solve the problem!*
- S2 : *So, based on the figure, $\frac{AF}{FB}$ is equal to $\frac{AFC}{BFC} - \frac{APF}{BPF}$ so $\frac{AF}{FB} = \frac{APC}{BPC}$. For $\frac{BD}{DC}$, it is $\frac{BDA}{CDA} - \frac{BPD}{CPD}$, so the result is $\frac{APB}{APC}$. Now the last one, $\frac{CE}{EA}$, that is from $\frac{CEB}{AEB} - \frac{CPE}{APE}$. So, the result is $\frac{BPC}{BPA}$.*
- R : *You mentioned that $\frac{AF}{FB} = \frac{APC}{BPC}$. Now at the beginning, it was mentioned that the ratio of the triangle area. Which one shows that the solution using the area ratio of a triangle?*
- S2 : *Oh that's right,. Can I repeat it?*
- R : *Yes, please.*
- S2 : *So $\frac{AF}{FB}$ it is obtained from the comparison $\frac{L\Delta AFC}{L\Delta BFC} - \frac{L\Delta APF}{L\Delta BPF}$. The result is $\frac{L\Delta APC}{L\Delta BPC}$. For $\frac{BD}{DC}$, it is obtained from $\frac{L\Delta BDA}{L\Delta CDA} - \frac{L\Delta BPD}{L\Delta CPD}$. The result is $\frac{L\Delta APB}{L\Delta APC}$. Now, for the last one $\frac{CE}{EA}$, it is obtained from $\frac{L\Delta CEB}{L\Delta AEB} - \frac{L\Delta CPE}{L\Delta APE}$. The result is $\frac{L\Delta BPC}{L\Delta BPA}$.*
- R : *Try to mention from the 1st equation to the 3rd equation!*
- S2 : *So, the first one is $\frac{AF}{FB} = \frac{L\Delta APC}{L\Delta BPC}$ the second $\frac{BD}{DC} = \frac{L\Delta APB}{L\Delta APC}$ the last one $\frac{CE}{EA} = \frac{L\Delta BPC}{L\Delta BPA}$.*
- R : *What is the result?*
- S2 : *So, we get $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{L\Delta APC}{L\Delta BPC} \cdot \frac{L\Delta APB}{L\Delta APC} \cdot \frac{L\Delta BPC}{L\Delta BPA}$. The result is 1. So, it can be concluded that $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.*

Explaining

Subject S2 initially appeared confused but recorded known information, reaching assimilation and equilibrium when identifying the problem's details accurately. This aligns with Swastika et al. (2023), stating that subjects often undergo an assimilation process while describing the source of a problem, as reflected in their ability to express the problem representation and identify known information verbally. Subsequently, S2 entered the accommodation process when a mistake was made in mentioning the question posed by the problem. This was evident when S2 paused momentarily, corrected himself, and then accurately restated the question. This finding supports Subanji (2021), who explained that individuals' adaptation occurs through assimilation and accommodation.

When S2 identified the theorem referenced in the question, he entered the assimilation stage, though this was accompanied by a state of disequilibrium. Following clarification by the researcher, S2 progressed to accommodation, correctly naming the theorem. Disequilibrium reemerged with confusion about the next step, but S2 transitioned back to accommodation, refining his approach by ultimately comparing congruent triangle areas. This iterative process of assimilation and accommodation led S2 to equilibrium, allowing him to arrive at the correct answer confidently. These observations align with Subanji (2011), who noted that during learning, individuals experience disequilibrium, with cycles of assimilation and accommodation continuing until equilibrium is achieved. Subject S2 also showed the assimilation stage when he successfully mentioned the height of the triangles and identified the congruent triangles. When S2 said “So, based on the figure, $\frac{AF}{FB}$ is equal to $\frac{AFC}{BFC} - \frac{APF}{BPF}$ so $\frac{AF}{FB} = \frac{APC}{BPC}$. For $\frac{BD}{DC}$, it is $\frac{BDA}{CDA} - \frac{BPD}{CPD}$, so the result is $\frac{APB}{APC}$. Now the last one, $\frac{CE}{EA}$ that is from $\frac{CEB}{AEB} - \frac{CPE}{APE}$. So, the result is $\frac{BPC}{BPA}$,” was incomplete so the researcher provided understanding. After receiving the explanation, S2 understood and explained coherently and experienced *equilibrium*.

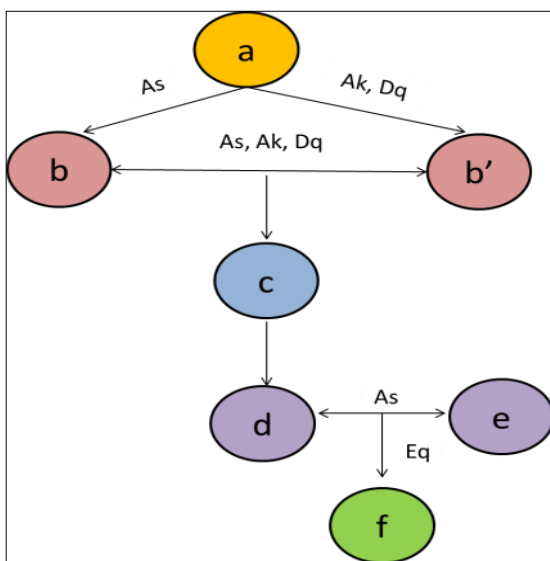


Figure 5. The thinking process of medium ability subject

Description:

a	: Test question to prove Ceva's theorem	f	: Conclusion
b	: Known in the question	As	: Assimilation
b'	: Asked in the question	Ak	: Accommodation
c	: Theorem to be proved	Eq	: Equilibrium
d	: State the height of a triangle	Dq	: Disequilibrium
e	: List the congruent triangles		

Conclusion

Research indicates that high-ability students, such as S1, seamlessly engaged in assimilation and accommodation without initial confusion when proving Ceva's Theorem, accurately identifying congruent triangles. S1 demonstrated exceptional fluency in explaining the problem-solving steps, leading to a correct answer and attaining the equilibrium stage. In contrast, medium-ability students exhibited some initial confusion when approaching the proof of Ceva's Theorem. Subject S2 recorded known information, indicating assimilation, but encountered disequilibrium due to an error in stating the question. After pausing and correcting himself, S2 omitted drawing the triangle. When mentioning the theorem associated with the problem, S2 again entered assimilation, accompanied by disequilibrium. Following clarification from the researcher, S2 transitioned to accommodation, correctly naming the theorem. S2 explained each solution step fluently, demonstrating confidence and reaching equilibrium after progressing through the stages of disequilibrium, assimilation, and accommodation.

Based on these findings, the following recommendations are proposed: 1) This research provides a valuable foundation for further exploration of students' thought processes in proving Ceva's Theorem, particularly considering diverse characteristics and materials. 2) Future studies should revisit the thought process involved in proving Ceva's Theorem with a larger and more varied sample of participants, 3) Addressing the limitations of this study is recommended to achieve more optimal results and enable observations of differences in the thinking processes between male and female students.

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