

On locating-chromatic number of helm graph H_m for $10 \leq m \leq 28$



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Abstract. Let $G = (V, E)$ be a connected graph and c be a k -coloring of G . The color class S_i of G is a set of vertices given color i , for $1 \leq i \leq k$. Let $\Pi = \{S_1, S_2, \dots, S_k\}$ be an ordered partition of $V(G)$. The color code of a vertex $v \in V(G)$ is defined as the ordered k -tuple $c_\Pi(v) = (d(v, S_1), d(v, S_2), \dots, d(v, S_k))$, where $d(v, S_i) = \min\{d(v, x) \mid x \in S_i\}$ for $1 \leq i \leq k$. If distinct vertices have distinct color codes, then c is called a locating-coloring of G . The locating-chromatic number $\chi_L(G)$ is the minimum number of colors in a locating-coloring of G . This paper discusses the locating-chromatic number of helm graph H_m for $10 \leq m \leq 28$. Helm graph H_m is constructed by adding some leaves to the corresponding vertices of wheels W_m , for $m \geq 3$.

Keywords: Locating-chromatic number, Wheel Graph, Helm Graph

INTRODUCTION

The concept of locating chromatic number was firstly introduced by Chartrand *et al* in 2002 [1]. Chartrand *et al* obtained the locating-chromatic number of several classes of graph, i.e cycle, path, and complete multipartite graph. They also proved that for any integer $k \in [3, n]$ and $k \neq n - 1$, there is always a tree of order n with the locating-chromatic number k [2]. Asmiati *et al* in 2011 determined the locating-chromatic number for amalgamation of stars [3]. Asmiati *et al* in 2012 determined the locating-chromatic number for firecracker graph [4]. In the same year, Asmiati & Baskoro were characterizing all graphs containing cycles with locating-chromatic number 3 [5]. Welyyanti *et al* showed the locating-chromatic number of complete n -ary trees [6]. Welyyanti *et al* determined the locating-chromatic number for graphs with dominant vertices [7]. Behtoei *et al* obtained the locating-chromatic number of the join graphs [8] Syofyan *et al* also determined the locating-chromatic number of homogeneous lobsters [9].

Welyyanti *et al* in 2014 expanded the meaning of the locating-chromatic number of a graph applied to all types of graphs including disconnected graphs [10]. Welyyanti *et al* determined the locating-chromatic number of disconnected graph with path and cycle graph as its components [11]. Welyyanti *et al* explain Some Conditions are Sufficient for Finite Locating-Chromatic Numbers on a Disconnected Graph [12]. Welyyanti *et al* has found on locating-chromatic number of disconnected graph with path, cycle, stars or double stars as its components [13]. Asmiati *et al* has studied the locating-

chromatic number of a disjoint union of some double stars [14]. Welyyanti *et al* determined the locating-chromatic number for graph with two homogenous components [15].

Rahayu and Kuswadi explained decomposition helm graph [16]. Lessya *et al* in 2023 has studied the locating-chromatic number helm graph H_m where $\chi_L(H_m) = 4$, for $3 \leq m \leq 9$ except for $m = 8$ has $\chi_L(H_m) = 5$ [17].

Let c be a k -coloring of a graph G . Let $\Pi = \{S_1, S_2, \dots, S_k\}$ be the partition of $V(G)$ induced by c , where S_i is the set of all vertices receiving color i . The color code $c_\Pi(v)$ of a vertex $v \in H$ is the ordered k -tuple $(d(v, S_1), d(v, S_2), \dots, d(v, S_k))$, where $d(v, S_i) = \min\{d(v, x) \mid x \in S_i\}$ for all $i \in [1, k]$. If all vertices of H have distinct color codes, then c is called a locating k -coloring of G , denoted by $\chi_L(G)$, is the smallest k such that H admits a locating-coloring with k colors. For any locating-coloring c of graph G , define a dominant vertex as a vertex v with $d(v, S_i) = 1$ for all color set S_i not containing v .

METHODOLOGY

In this research, we are collecting literature related to helm graph H_m , learning the theory of locating-chromatic number theory which supports the determination of locating-chromatic number of helm graph H_m , and analyzing the characteristics of helm graph. We studying locating-chromatic number of helm graph H_m with $10 \leq m \leq 28$.



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RESULTS AND DISCUSSION

A circle graph is a simple connected graph with m vertices with each vertex of degree two, denoted by C_m . A wheel graph is a connected graph constructed from a circle graph by adding a center vertex x and m edges such that vertex x neighbors all the points in the circle graph, denoted by W_m . The helm graph is constructed by adding point y_i , with $1 \leq y_i \leq m$, which is adjacent to vertex x_i in the wheel graph W_m . In Figure 1 the H_m helm graph is given as follows.

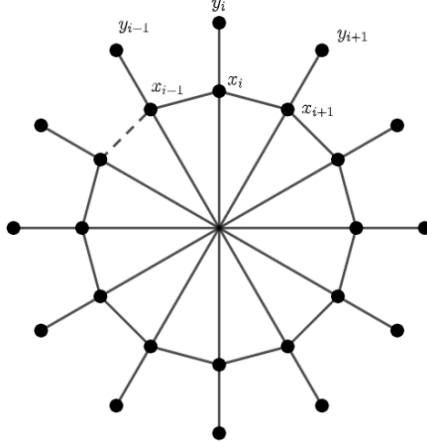


Figure 1. Helm graph

In Theorem 1, we will determine the locating-chromatic number of the helm graph H_m for $10 \leq m \leq 28$.

Theorem 1. Let H_m be a helm graph for $10 \leq m \leq 28$. Then $\chi_L(H_m) = 5$.

Proof. Suppose there is a helm graph H_m with the following set of vertices and edges:

$$\begin{aligned} V(H_m) &= \{x_0\} \cup \{x_i, y_i \mid 1 \leq i \leq m\}, \\ E(H_m) &= \{x_0 x_i, x_i y_i \mid 1 \leq i \leq m\} \cup \{x_j x_{j+1} \mid 1 \leq j \leq m-1\} \cup \{x_1 x_m\}. \end{aligned}$$

Define vertex x_0 as the center of the graph H_m , and define $N(x_i) = \{x_0, x_{i-1}, x_{i+1}, y_i\}$ as the set of neighbors of x_i . We will prove the upper and lower bounds of the locating-chromatic number for $10 \leq m \leq 28$. Next, it will be shown that for all possibilities of 4-coloring of H_m , we always have at least two vertices with the same color codes. Without loss of generality, let $c(x_0) = c(y_i) = 1$. All possibilities of 4-locating coloring obtained are given in Table 1.

Table 1. Locating 4-coloring graph for $c(x_0) = c(y_i) = 1$

No	$c(x_{i-1})$	$c(x_i)$	$c(x_{i+1})$
1	2	3	2
2	2	3	4
3	2	4	2
4	2	4	3
5	3	2	3
6	3	2	4
7	3	4	3
8	4	2	4
9	4	3	4

Based on Table 1, there are 9 possibilities for 4-coloring of H_m . Then number of possible locating coloring graph is m . If we add one or more color to vertex from Table 1, then we have at least two vertices with the same color codes. So, that we obtain $\chi_L(H_m) \geq 5$ for $m \geq 10$. For $c(x_0) \neq c(y_i) \neq 1$, let H_m has a coloring vertex with four colors. That coloring vertex of H_m will have two vertices dominant which have same color. It means that two dominant vertices have same color codes. So, $\chi_L(H_m) \geq 5$ for $m \geq 10$.

It will be shown that $\chi_L(H_{10}) \leq 5$. Define $c : V(H_{10}) \rightarrow \{1, 2, 3, 4, 5\}$ and $\Pi = \{S_1, S_2, S_3, S_4, S_5\}$ as follows:

$$\begin{aligned} S_1 &= \{x_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}, \\ S_2 &= \{x_1\}, \\ S_3 &= \{x_2, x_4, x_9\}, \\ S_4 &= \{x_6, x_8, x_{10}\}, \\ S_5 &= \{x_3, x_5, x_7\}. \end{aligned}$$

The color codes for all vertices of H_{10} are as follows:

$$\begin{aligned} c_\Pi(x_0) &= (0, 1, 1, 1, 1), c_\Pi(x_1) \\ &= (1, 0, 1, 1, 2), c_\Pi(x_2) \\ &= (1, 1, 0, 2, 1), \\ c_\Pi(x_3) &= (1, 2, 1, 2, 0), c_\Pi(x_4) \\ &= (1, 2, 0, 2, 1), c_\Pi(x_5) \\ &= (1, 2, 1, 1, 0), \\ c_\Pi(x_6) &= (1, 2, 2, 0, 1), c_\Pi(x_7) \\ &= (1, 2, 2, 1, 0), c_\Pi(x_8) \\ &= (1, 2, 1, 0, 1), \\ c_\Pi(x_9) &= (1, 2, 0, 1, 2), c_\Pi(x_{10}) \\ &= (1, 1, 1, 0, 2), c_\Pi(y_1) \\ &= (0, 1, 2, 2, 3), \\ c_\Pi(y_2) &= (0, 2, 1, 3, 2), c_\Pi(y_3) \\ &= (0, 3, 2, 3, 1), c_\Pi(y_4) \\ &= (0, 3, 1, 3, 2), \\ c_\Pi(y_5) &= (0, 3, 2, 2, 1), c_\Pi(y_6) \\ &= (0, 3, 3, 1, 2), c_\Pi(y_7) \\ &= (0, 3, 3, 2, 1), \\ c_\Pi(y_8) &= (0, 3, 2, 1, 2), c_\Pi(y_9) \\ &= (0, 3, 1, 2, 3), c_\Pi(y_{10}) \\ &= (0, 2, 2, 1, 3). \end{aligned}$$

Because every two vertices have different color codes then $\chi_L(H_{10}) \leq 5$. Next, we will show that shown $\chi_L(H_{28}) \leq 5$. Define $c : V(H_m) \rightarrow \{1, 2, 3, 4, 5\}$ and $\Pi = \{S_1, S_2, S_3, S_4, S_5\}$ as follows:

$$\begin{aligned} S_1 &= \{x_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, \\ & y_{14}, y_{18}, y_{19}, y_{20}, y_{21}, y_{22}, y_{23}, y_{24}\}, \\ S_2 &= \{x_{12}, x_{14}, x_{17}, x_{19}, x_{21}, x_{25}, x_{11}, y_{27}\}, \\ S_3 &= \{x_2, x_4, x_9, x_{11}, x_{13}, x_{23}, x_{26}, y_{28}\}, \\ S_4 &= \{x_6, x_8, x_{10}, x_{15}, x_{20}, x_{22}, x_{26}, x_{28}\}, \\ S_5 &= \{x_3, x_5, x_7, x_{16}, x_{18}, x_{24}, x_{27}, y_{1}\}. \end{aligned}$$

The color codes for all vertices of H_{28} are as follows:

$$\begin{aligned} c_\Pi(x_0) &= (0, 1, 1, 1, 1), c_\Pi(x_1) \\ &= (1, 0, 1, 1, 1), c_\Pi(x_2) \\ &= (1, 1, 0, 2, 1), \end{aligned}$$

$$\begin{aligned}
 c_{\Pi}(x_3) &= (1, 2, 1, 2, 0), c_{\Pi}(x_4) \\
 &= (1, 2, 0, 2, 1), c_{\Pi}(x_5) \\
 &= (1, 2, 1, 1, 0), \\
 c_{\Pi}(x_6) &= (1, 2, 2, 0, 1), c_{\Pi}(x_7) \\
 &= (1, 2, 2, 1, 0), c_{\Pi}(x_8) \\
 &= (1, 2, 1, 0, 1), \\
 c_{\Pi}(x_9) &= (1, 2, 0, 1, 2), c_{\Pi}(x_{10}) \\
 &= (1, 2, 1, 0, 2), c_{\Pi}(x_{11}) \\
 &= (1, 1, 0, 1, 2), \\
 c_{\Pi}(x_{12}) &= (1, 0, 1, 2, 2), c_{\Pi}(x_{13}) \\
 &= (1, 1, 0, 2, 2), c_{\Pi}(x_{14}) \\
 &= (1, 0, 1, 1, 2), \\
 c_{\Pi}(x_{15}) &= (1, 1, 2, 0, 1), c_{\Pi}(x_{16}) \\
 &= (1, 1, 2, 1, 0), c_{\Pi}(x_{17}) \\
 &= (1, 0, 2, 2, 1), \\
 c_{\Pi}(x_{18}) &= (1, 1, 2, 2, 0), c_{\Pi}(x_{19}) \\
 &= (1, 0, 2, 1, 1), c_{\Pi}(x_{20}) \\
 &= (1, 1, 2, 0, 2), \\
 c_{\Pi}(x_{21}) &= (1, 0, 2, 1, 2), c_{\Pi}(x_{22}) \\
 &= (1, 1, 1, 0, 2), c_{\Pi}(x_{23}) \\
 &= (1, 2, 0, 1, 1), \\
 c_{\Pi}(x_{24}) &= (1, 1, 1, 2, 0), c_{\Pi}(x_{25}) \\
 &= (1, 0, 1, 2, 1), c_{\Pi}(x_{26}) \\
 &= (1, 1, 0, 1, 1), \\
 c_{\Pi}(x_{27}) &= (1, 1, 1, 1, 0), c_{\Pi}(x_{28}) \\
 &= (1, 1, 1, 0, 1), c_{\Pi}(y_1) \\
 &= (2, 1, 2, 2, 0), \\
 c_{\Pi}(y_5) &= (0, 3, 2, 2, 1), c_{\Pi}(y_6) \\
 &= (0, 3, 3, 1, 2), c_{\Pi}(x_7) \\
 &= (0, 3, 3, 2, 1), \\
 c_{\Pi}(x_8) &= (0, 3, 2, 1, 2), c_{\Pi}(x_9) \\
 &= (0, 3, 1, 2, 3), c_{\Pi}(x_{10}) \\
 &= (0, 3, 2, 1, 3), \\
 c_{\Pi}(x_{11}) &= (0, 2, 1, 2, 3), c_{\Pi}(x_{12}) \\
 &= (0, 1, 2, 3, 3), c_{\Pi}(x_{13}) \\
 &= (0, 2, 1, 3, 3), \\
 c_{\Pi}(x_{14}) &= (0, 1, 2, 2, 3), c_{\Pi}(x_{15}) \\
 &= (0, 2, 3, 1, 2), c_{\Pi}(x_{16}) \\
 &= (0, 2, 3, 2, 1), \\
 c_{\Pi}(x_{17}) &= (0, 1, 3, 3, 2), c_{\Pi}(x_{18}) \\
 &= (0, 2, 3, 3, 1), c_{\Pi}(x_{19}) \\
 &= (0, 1, 3, 2, 2), \\
 c_{\Pi}(x_{20}) &= (0, 2, 3, 1, 3), c_{\Pi}(x_{21}) \\
 &= (0, 1, 3, 2, 3), c_{\Pi}(x_{22}) \\
 &= (0, 2, 2, 1, 3), \\
 c_{\Pi}(x_{23}) &= (0, 3, 1, 2, 2), c_{\Pi}(x_{24}) \\
 &= (0, 2, 2, 3, 1), c_{\Pi}(x_{25}) \\
 &= (0, 1, 2, 3, 2), \\
 c_{\Pi}(x_{26}) &= (2, 2, 1, 0, 2), c_{\Pi}(x_{27}) \\
 &= (2, 0, 2, 2, 1), c_{\Pi}(x_{28}) \\
 &= (2, 1, 2, 2, 0), \\
 c_{\Pi}(y_2) &= (0, 2, 1, 3, 2), c_{\Pi}(y_3) \\
 &= (0, 3, 2, 3, 1), c_{\Pi}(y_4) \\
 &= (0, 3, 1, 3, 2).
 \end{aligned}$$

Because every two vertices have different color codes then $\chi_L(H_{28}) \leq 5$.

Let $c: V(H_m) \rightarrow \{1, 2, 3, 4, 5\}$ where $c(x_0) = 1$. In following table, we define the vertex coloring for other vertices.

Table 2. The vertex coloring graph $c: V(H_m) \rightarrow \{1, 2, 3, 4, 5\}$ for $1 \leq i \leq m, m \geq 10$

i	$c(x_{i-1})$	$c(x_i)$	$c(x_{i+1})$	$c(y_i)$
2	2	3	5	1
3	3	5	3	1
4	5	3	5	1
5	3	5	4	1
6	5	4	5	1
7	4	5	4	1
8	5	4	3	1
9	4	3	4	1
10	3	4	3	1
11	4	3	2	1
12	3	2	3	1
13	2	3	2	1
14	3	2	4	1
15	2	4	5	1
16	4	5	2	1
17	5	2	5	1
18	2	5	2	1
19	5	2	4	1
20	2	4	2	1
21	4	2	4	1
22	2	4	3	1
23	4	3	5	1
24	3	5	2	1
25	5	2	3	1
26	2	3	5	4
27	3	5	4	2
28	5	4	2	3
1	4	2	3	5

In Table 2, if $i = 1$ then $c(x_{i-1}) = c(x_m)$ and if $i = 28$ then $c(x_{i+1}) = c(x_1)$. Example, for $m = 11$, it can use $i = 2, 3, 4, 5, 6, 7, 8, 9, 10$ for color vertex in H_{11} (Table 2).

Based on Table 2, it is obtained locating-coloring graph for $10 \leq m \leq 28$ resulting from Case 1 and Case 2, so that each vertex with $10 \leq m \leq 28$ has a different color code. If we add one or more color to vertex from Table 4, then we have at least two vertices with the same color codes. So that the locating-chromatic number for the helm graph H_m for $10 \leq m \leq 28$ is $\chi_L(H_m) = 5$.

CONCLUSION

Let H_m be a helm graph for $10 \leq m \leq 28$. This paper determines that the locating-chromatic number of H_m is 5, or $\chi_L(H_m) = 5$, for $10 \leq m \leq 28$.

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