

Analysis of particle dynamics, event horizon, and thermodynamic properties of black holes in the *Ghasemi-Nodehi-Bambi metric*



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Abstract. Black holes, as complex astrophysical objects, are strongly influenced by their metric parameters. This research focuses on the Ghasemi-Nodehi-Bambi (G-N-B) metric, an extension of the Kerr solution, which introduces additional parameters to more accurately describe the properties of black holes. The research investigates the impact of parameter variations in the G-N-B metric on black hole thermodynamics, particle dynamics, and event horizon structure. Analytical and numerical methods are applied to examine the quadratic equations governing event horizons, particle motion, and black hole temperature and entropy. The research explores the variation of parameters (M) and (a) within a certain range to evaluate their influence on the gravitational distribution and the effect of frame attraction. The distribution of (M) values was visualized on a logarithmic scale to highlight the sensitivity of the system to parameter changes. The study found that variations in the G-N-B metric parameter significantly affect the event horizon, with the likelihood of extreme black holes or naked singularities forming depending on the discriminant of the quadratic equation. Particle motion is affected by parameters (M) and (a), which alter the gravitational field and orbital stability. The black hole temperature and entropy show significant changes: an increase in (M) increases gravity and surface temperature, while an increase in (a) decreases temperature due to rotational effects. The research improves the understanding of black holes beyond the Kerr model, especially in terms of black hole thermodynamics and time-space structure.

Keywords: Black Hole Thermodynamics, *Ghasemi-Nodehi-Bambi metric*, Gravitational Modeling, Non-linear Behavior, Parameter Variations

INTRODUCTION

The space-time metric of the general theory of relativity became an important pillar in the development of modern theoretical physics [1]. One metric that has attracted attention is the *Ghasemi-Nodehi-Bambi metric*, which is designed to understand the nature of spacetime [2], [3], [4], [5], [6]. In general relativity theory, tests have been conducted on weak gravitational fields through experiments in the Solar System and observations of binary pulsars [7], [8]. More recently, attention has shifted to testing this theory under more extreme conditions, such as verifying the Kerr properties of astrophysical black hole candidates [9], [10]. In the framework of general relativity, the spacetime geometry around astrophysical black holes should be described by

Kerr solutions [11]. Testing the *Schwarzschild metric* in the weak field, one can write a static, spherically symmetric, and asymptotically flat metric with expansion in (M/r) , where (M) is the mass of the central object and (r) is the radial coordinate [12], [13]. Testing in strong gravitational fields is complicated because the (M/r) ratio is no longer small, resulting in many deviations from the *Kerr metric*. To overcome this, a common approach is to use a metric with a number of "deformation parameters", where a zero inequality in these parameters indicates a deviation from the Kerr geometry [14], [15]. The *Ghasemi-Nodehi-Bambi metric* takes a different approach by introducing 11 Kerr parameters, all of which are 1 in the standard *Kerr metric*. If the parameter values deviate from 1, this



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indicates that the mass and momentum of the spin change the geometry of space-time more or less than general relativity predicts. The approach verifies the Kerr solution, rather than finding a new form, by applying this parametrization to the shadow of a black hole that can be observed in the future [16]. As computing technology advances, there is a need to model parameters more accurately. The effect of polar angle on the angular momentum distribution and the relationship of other parameters to the metric value still require in-depth exploration [17].

The study of spacetime metrics began with Albert Einstein's monumental work in 1915, the general theory of relativity [18]. In theory, the concept of space-time is introduced as a four-dimensional manifold involving the interaction between mass and energy [19]. In 1963, Roy P. Kerr introduced the Kerr solution, which was the first solution to general relativity that included angular momentum (spin) [20], [21]. Many metric solutions have been developed to explain various gravitational phenomena. One of the relevant metrics is the *Ghasemi-Nodehi-Bambi (GNB)* metric, which was introduced to deal with complexities that cannot be described by the Kerr solution. The metric allows testing against small deviations from the Kerr solution by changing the free parameters (b_i); when all ($b_i = 1$), the metric reverts to the *Kerr metric*. The main focus of the metric is to identify observable parts of space-time and test for small deviations through observations, so it can be used as an alternative to test alternative theories of gravity with constraints that match observations in the solar system [22]. In the 21st century, advances in computing technology allow for the simulation and numerical analysis of more complex metrics [23]. The parameters in the metric can be analyzed on a large scale to understand their distribution in various astrophysical scenarios.

The *Ghasemi-Nodehi-Bambi metric* is one of the solutions in general relativity theory that describes the space-time properties around extreme astrophysical objects such as black holes. The metric introduces several physics parameters, including the a parameter [2]. One interesting aspect of the *Ghasemi-Nodehi-Bambi metric* is its dependence on parameters that exhibit complex non-linear properties, especially the rotation parameter a . The value of a has a large effect on the response of the system to variations in mass, radius, and angle. At high values of a , the system shows a greater response to small changes in these physical parameters. As for the angle θ , through the function $\cos^2(\theta)$, it also affects the results of metric calculations, especially at extreme angles such as 0° and 180° , where the value of a tends to be small and affects the results of mass and radius. In contrast, at intermediate angles such as 90° , the value of a tends to be larger.

This study aims to investigate the highly sensitive non-linear relationship between the parameters in the *Ghasemi-Nodehi-Bambi metric*, such as mass (M), radius (r), and angle (θ). It is expected that small changes in

any of the parameters will cause significant fluctuations in the calculation results of the metric, especially when the parameters r and θ are within a certain range. This non-linear relationship can be the basis for knowing how the space-time system responds to changes in the physical variables involved.

The hypothesis is based on exploring the physics parameters in the *Ghasemi-Nodehi-Bambi metric* to understand the deformation of the space-time geometry around black holes. The first hypothesis states that the larger the value of the rotation parameter (a), the wider the range of values of the metric (M), which interprets the increasing fluctuations in the geometry of spacetime due to the strong influence of the black hole rotation [24]. The second hypothesis suggests that the parameters b_1 , b_2 , and the polar angle (θ) affect the distribution of the M parameter values. In particular, larger a values tend to appear at angles close to 90° or 270° , indicating high sensitivity to the polar position around the black hole. The relationship between r , a , and θ is expected to produce a non-linear effect in the M metric distribution, where small variations in the a parameter can lead to significant changes in the M value. An additional hypothesis reinforces this observation by stating that when ($b_1 \approx 1$), the value of M approaches the *Kerr metric* solution. Under asymptotic conditions ($r \rightarrow \infty$), the predicted metric deformation remains consistent with general relativity, where ($b_1 M$) represents the total mass of the system. A three dimensional (3D) representation of the relationship between M , r , and θ against a will show a systematic pattern that can identify the dominant parameter in the space-time deformation. The physical hypothesis confirms that the interaction between M and a values result in complex gravitational effects, with higher M values indicating stronger gravitational interactions and more significant spacetime deformations around the black hole.

The study has a number of limitations, mainly related to the focus on the dt^2 component in the modified *Ghasemi-Nodehi-Bambi metric* for black holes. The study considers deformations of the space-time geometry due to small changes in the b_i parameter and does not explore contributions from other alternative theories of gravity. Therefore, the results obtained are limited to black holes close to the Kerr solution, with relatively small deformations. The analysis is restricted to a certain range of a and r parameter values, so the conclusions cannot be applied to larger or smaller parameter ranges. The study only includes parameters directly related to the geometry of space-time, specifically those affecting the dt^2 component, without considering other external factors such as additional gravitational fields or interactions with matter around the black hole. In the visualization, the 3D model used only includes the relationship between the parameters M , r , and θ against a , without depicting more complex space-time configurations. The approach used in the research is based on numerical calculations and descriptive statistics, which have limitations in capturing small or

extreme fluctuations in parameter distributions. The research is limited to a certain number of parameters, while the influence of other parameters that can affect the dt^2 metric is not analyzed in depth. Further testing, especially with more detailed observational data, is required to validate the results obtained in real-world observations.

The research fills a gap in previous studies on black hole metric deformation, particularly in the influence of additional parameters on the *Ghasemi-Nodehi-Bambi metric*. Most of the previous studies have focused on the *Kerr metric* as the standard model of space-time geometry around black holes. However, studies on the deformation of the *Kerr metric* involving additional parameters, such as the *Ghasemi-Nodehi-Bambi metric*, are still very limited. One of the major shortcomings in previous studies is the lack of in-depth analysis of how the b_i parameter affects the distribution of the a parameter and the M metric, which is crucial for describing fluctuations in the space-time geometry around black holes. Most previous studies only offer mathematical analysis without including a clearer 3D visualization of the complex relationship between parameters M , r , a , and angle (θ). In fact, this visualization is important to obtain a more in-depth picture of the space-time deformation pattern. Previous studies have not explored the effect of non-linearity between the parameters r , a , and θ on the metric distribution M . In fact, complex non-linear relationships play a role in a more accurate understanding of the space-time phenomena occurring around black holes. The research contributes by introducing a clearer visualization and a more comprehensive analysis, revealing the dynamical interactions between these parameters in the *Ghasemi-Nodehi-Bambi metric*, which has not previously received enough attention [25]. As for general relativity, it has become the main reference in space-time geometry research, His research in *Ghasemi-Nodehi-Bambi metric* deformations, especially under asymptotic conditions, is still limited.

The study of the Ghasemi-Nodehi-Bambi (G-N-B) metric has become an important research topic in understanding the space-time dynamics around black holes. The metric was developed as an alternative to the *Kerr metric*, which has been the standard model in general relativity theory to describe rotating black holes. One of the main motivations behind developing the G-N-B metric was to test the Kerr hypothesis and provide a broader parameterization of the strong gravitational effects around black holes. Several studies have focused on testing the parameters in the G-N-B metric using various observational approaches. [2] explored the parameters of this metric through quasi-periodic oscillations (QPOs), which are periodic oscillations in the X-ray emission from the accretion disk around a black hole. The study shows that the parameters in the G-N-B metric can provide a richer picture than the Kerr model in explaining the QPOs phenomenon, potentially becoming an important tool to test the properties of black

holes more precisely. In another study, [3] They tested this metric using the continuum-fitting method, which compares the radiation spectrum of the accretion disk with theoretical predictions to obtain spatial parameter constraints. The study emphasizes that the G-N-B metric can serve as a possible alternative in describing the space-time geometry around black holes, especially when relativistic effects are more complex than Kerr's predictions. The study conducted [4] conducted research on reverberation mapping, which was done to understand how light from accreted material around a black hole is altered by the effects of gravity. The research also highlighted how parameter variations in the G-N-B metric can be linked to the spectral characteristics of iron lines formed in the accretion disk.

Research conducted by [5] indicates that the metric can be seen as a modification of the extended Kerr geometry with quintessential fields, which in turn impacts the spectral line profile as well as the silhouette of the black hole shadow. This reinforces the idea that non-*Kerr metrics* can more accurately represent astrophysical observations that deviate from the predictions of classical general relativity. On the other hand, in a phenomenological approach, [26] discusses various possible non-*Kerr metrics* in tests of strong gravity using black hole observations. In the book *Black Holes: A Laboratory for Testing Strong Gravity*, the book highlights that metrics like G-N-B can provide a theoretical alternative in interpreting observational data that does not fully match Kerr's predictions. The review from the book also confirms that in certain scenarios, modified *Kerr metrics* such as G-N-B can better explain the observational data compared to the standard model of general relativity.

METHODOLOGY

Kerr Model Review

The *Kerr metric* is the solution to Einstein's field equations for an uncharged rotating black hole [27]. This solution is a generalization of the *Schwarzschild metric*, which only applies to black holes without rotation. The *Schwarzschild metric* equation in spherical coordinates (t, r, θ, ϕ) is given by [28]:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (1)$$

This solution applies to black holes without angular momentum. To include rotational effects, the Kerr-Schild ansatz method is utilized, which transforms the metric by adding a component of angular momentum (a), which is related to angular momentum (J) through the relation [29], [30]:

$$a = \frac{J}{M} \quad (2)$$

In Boyer-Lindquist coordinates, the *Kerr metric* is given by Eqs. (3) [31]. As seen in the equation which is the *Kerr metric*, the metric functions are given by ($\Sigma = r^2 + a^2 \cos^2 \theta$) and ($\Delta = r^2 - 2Mr + a^2$), where M is the mass of the black hole and a is the spin parameter which is the rotation of the black hole. To derive the *Kerr metric*, one starts with the *Schwarzschild metric* and transforms to a rotating coordinate system by replacing r with a new combination involving a . The coordinate transformation makes it possible to incorporate angular momentum into the metric, which changes the shape of the metric [31].

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2a^2Mr \sin^2 \theta}{\Sigma}\right)\sin^2 \theta d\phi^2 - \frac{4aMr \sin^2 \theta}{\Sigma}dtd\phi \quad (3)$$

After the coordinate change, two new functions are defined, namely the modified radial distance function (Σ) and the horizon function (Δ). The time-time component (g_{tt}) changes due to rotation, resulting in a frame-dragging effect. The space-time component ($g_{t\phi}$) arises due to the interaction between rotation and gravity, while the radial-space component (g_{rr}) and angular component ($g_{\theta\theta}$) are also affected by these changes.

By obtaining the above metric elements, it can be verified that the metric satisfies Einstein's field equations, with ($\Lambda = 0$) for the case of a Kerr black hole in vacuum. The event horizon in the *Kerr metric* is obtained by finding the radial value (r) that causes divergence in the metric component (g_{rr}). In order for ($g_{rr} \rightarrow \infty$), the denominator must be zero, i.e. ($\Delta = 0$). The Δ function for the *Kerr metric* is ($r^2 - 2Mr + a^2$), which can be solved using the quadratic formula to obtain two values of r , namely the Outer Horizon (Event Horizon) and the Inner Horizon (Cauchy Horizon).

The outer horizon is the limit where no object, including light, can escape the gravitational pull of a black hole. The inner horizon is more relevant in the context of instability due to quantum effects [32]. For the solution to ($r_{\pm}$) to be a real number, the discriminant must be non-negative, which means ($M^2 \geq a^2$). If ($M = a$), then the outer and inner horizons converge, resulting in only one horizon referred to as a Kerr extremal black hole.

If ($a > M$), then the discriminant value becomes negative, which means there is no event horizon and the singularity will be visible, which contradicts the Cosmic Censorship Hypothesis which states that the singularity must always be hidden in the horizon [33], [34].

Review of Ghasemi-Nodehi-Bambi metric with b_i Parameters to Improve Black Hole Geometry Deformation Model

The metric used in this study adapts the *Kerr metric* in Boyer-Lindquist coordinates by including the deformation parameter b_i , which modifies the space-time geometry around the rotating black hole. The line elements of the metric are rewritten to include the influence of mass, rotation, and other factors that distort space-time [35]. Deviations from the value $b_i = 1$ provide insight into unexpected effects on the structure of spacetime, potentially improving or changing the interpretation of the black hole phenomenon in general relativity. As for Boyer-Lindquist coordinates, the *Kerr metric* line element can be written as [36]:

$$ds^2 = \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2Mr + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + \left(\frac{r^2 + a^2 + 2Ma^2r \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}\right) \sin^2 \theta d\phi^2 - \left\{\left(1 - \frac{2M}{r}\right)(r^2 + a^2 \cos^2 \theta) dt^2 + \frac{4Mar \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} dtd\phi\right\} \quad (4)$$

The metric of Eqs. (4) interprets the space-time geometry around a rotating black hole, with r as the radial distance, θ as the polar angle, and ϕ as the azimuthal angle. The parameter a represents the angular momentum, and M is the mass of the black hole. The temporal component (dt^2) interprets the gravitational effect due to the mass and rotation of the black hole, while the cross component ($dtd\phi$) shows the effect of rotation on time and azimuthal angle. The (dr^2) component describes the radial distance effect, and ($d\theta^2$) and ($d\phi^2$) describe the space-time distortion in the vertical and azimuthal directions. The parameters r and a affect the orbital shift of the particle. An event horizon exists if ($|a| \leq M$), but if ($|a| > M$), the *Kerr metric* describes a horizonless spacetime and an open singularity.

In the Boyer-Lindquist metric, if mass ($M = 0$) and angular momentum ($a = 0$), the geometry of space-time becomes flat, describing a space without gravity and rotation [37]. To understand the effect of mass and rotation on space-time, parameters b_i are required that govern the effect of both. Each of these parameters modifies a component in the metric, affecting the distortion of space-time caused by mass (M) and rotation (a). The first component governs the time distortion due to the mass and rotation of the black hole, while the cross component ($dtd\phi$) describes the interaction between time and the azimuthal angle affected by the angular momentum (a). The changes in radial distance and vertical angle describe how space-time is distorted along the radial and vertical axes by the presence of the rotating black hole [38]. In the *Kerr metric*, $b_i = 1$ for all i , and if $b_i \neq 1$, then the associated mass or rotation distorts the geometry of space-time differently from the predictions of general relativity. The rigidity of space-time is described by the Riemann tensor, and the Kretschmann

scalar-like curvature invariant (K), which is defined as [39], [40]:

$$K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \quad (5)$$

For a generalized metric with parameter b_i , K can be expanded to:

$$K = K_{\text{Kerr}} + \sum_{i=1}^{11} K_i^{(1)} X_i (b_i - 1) + \dots, \quad (6)$$

where X_i is a combination of M , a , and the square or product between the two. This expansion describes deviations in the parameter $b_i \neq 1$ modifying the curvature of spacetime, which tests whether mass and spin can distort spacetime according to the predictions of general relativity. The parameter b_1 is crucial for determining the contribution of mass in the system, and in the Newtonian constraint, its value must be 1 to conform to Newton's law of gravity [41], [42]. When $b_1=1$, the solution obtained is in accordance with Newton's theory of gravity, which is relevant for objects with large masses such as black holes. Parameter b_6 affects radial gravity and horizon radius, which require values close to 1 in order to meet the precision of measurements in the Solar System (about 10^{-5}). Parameter b_3 governs the Lense-Thirring effect, which is observed with 10% accuracy, so the value of this parameter must be close to 1 to match the available data, especially on the movement of satellites and celestial bodies [43], [44]. Other parameters, such as b_2 , b_4 , and others, currently have no clear observational constraints, and their impact on space-time deformation is still theoretical, with the possibility of further testing as observational technology advances.

The *Ghasemi-Nodehi-Bambi metric* are as follows:

$$ds^2 = \frac{r^2 + b_5 a^2 \cos^2 \theta}{r^2 - 2b_6 M r + b_7 a^2} dr^2 - \left[1 - \frac{2b_1 M r}{r^2 + b_2 a^2 \cos^2 \theta} \right] dt^2 \\ + \frac{4b_3 M a r \sin^2 \theta}{r^2 + b_4 a^2 \cos^2 \theta} dt d\phi \\ + (r^2 + b_8 a^2 \cos^2 \theta) d\theta^2 \\ + \left(r^2 + b_9 a^2 \right. \\ \left. + \frac{2b_{10} M a r \sin^2 \theta}{r^2 + b_{11} a^2 \cos^2 \theta} \right) \sin^2 \theta d\phi^2 \quad (7)$$

Metric in Eqs. (7) describes the physical structure of the black hole gravitational field. The g_{tt} component represents the gravitational potential, with b_1 determining the contribution of the monopole mass and b_2 including the influence of the angular momentum distribution ($a^2 \cos^2 \theta$). If b_1 and b_2 deviate from general relativity, the shape of the black hole shadow will change. The $g_{t\phi}$ component describes the frame-dragging effect due to the rotation of the black hole, with b_3 governing the intensity of the interaction between angular momentum and ambient energy. The g_{rr} component represents the influence of radial gravity,

with b_5 , b_6 , and b_7 affecting the horizon radius and the surrounding gravitational behavior. The $g_{\theta\theta}$ component is the deformation of polar geometry, with b_8 affecting the unsymmetrical distribution of energy or matter. The $g_{\phi\phi}$ component is related to azimuthal trajectories, where b_9 , b_{10} , and b_{11} affect the symmetry of photon trajectories, which can change the shape of the black hole shadow to be flatter or asymmetric. The line elements of the *Kerr metric* can be recovered when all $b_i = 1$. The asymptotic mass is $b_1 M$, and the asymptotic specific angular momentum is $b_3 a$ {if ($b_1 = 1$)}. Based on experiments in the Solar System, b_6 is very close to 1, so b_1 , b_3 , and b_6 can be considered equal to 1. Meanwhile, the other eight parameters b_2 , b_4 , b_5 , b_7 , b_8 , b_9 , b_{10} , and b_{11} are treated as free parameters. This metric is a general description of black holes in alternative theories of gravity, and assumes that the metric around the black hole is close to the Kerr solution, with analysis to limit small deviations from that solution.

RESULTS AND DISCUSSION

Kerr metric Analysis

Outer horizon (r_+) marks the boundary where nothing, including light, can escape. The inner horizon (r_-) is the causal limit inside the black hole. Given $r_{\pm}$, the mass M and spin parameter (a) can be determined as:

$$M = \frac{r_+ + r_-}{2} \quad (8)$$

$$a = \sqrt{M^2 - \left(\frac{r_+ - r_-}{2} \right)^2} \quad (9)$$

If only r_+ is known:

$$M = \frac{r_+^2 + a^2}{2r_+} \quad (10)$$

$$a = \sqrt{r_+^2 - 2Mr_+} \quad (11)$$

Let $\rho^2 = r^2 + a^2 \cos^2 \theta$, then:

$$\cos^2 \theta = \frac{\rho^2 - r^2}{a^2} \quad (12)$$

The relationship between black hole mass (M), event horizon radius (r_+), and spin parameter (a) in *Kerr metric* is studied. The value of r_+ is varied from 1.1 to 5 in steps of 0.1. The spin parameter (a) is tested at three scenarios, namely 0.1, 0.5, and 1.0, to explore how the increase in spin affects the black hole mass.

From the analysis, it is seen that as the value of the spin parameter (a) increases, the mean and median mass (M) also increases. For $a = 0.1$, the minimum mass obtained is about 0.555, while the average is about 1.53. When a increases to 0.5, the minimum mass increases to 0.664, with an average of 1.57. At the extreme value of $a = 1$, the minimum mass reaches 1.00, and the average increases to about 1.72. This shows that the larger the value of a , the larger the mass of the resulting black hole for the same event horizon.

The standard deviation (SD) of mass tends to decrease as the value of a increases, which means that the mass

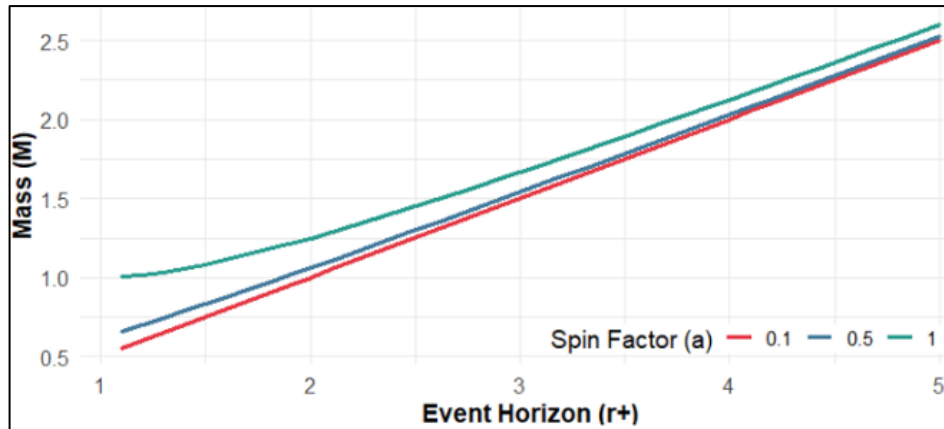


Figure 1. the relationship between mass (M) and event horizon (r_+) for various spin factors in the *Kerr metric*

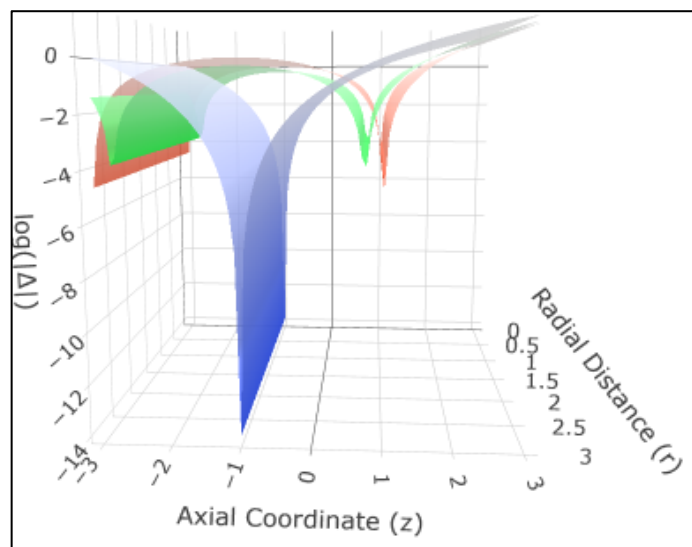


Figure 2. Variation of Spin Parameter against Horizon Structure in *Kerr metric*

distribution is narrower and less varied at larger values of a compared to smaller values of a . This phenomenon can be explained through the basic form of the equation which shows that the contribution of a^2 becomes more significant in the calculation of mass (M) as the value of a increases.

The increased spin leads to stronger relativistic gravitational effects, which in turn affects the radius of the event horizon as well as the structure of space-time around the black hole. The results of this study are in line with the theoretical prediction that black holes with higher spins will have a more centralized and less variable mass distribution compared to black holes with almost no spin [45].

Physically, Fig. 1 shows the relationship between mass (M) and the event horizon (r_+) in the *Kerr metric* for various values of spin (a). The different colored curves represent different values of a , and it appears that the larger the value of a , the larger the mass (M) is required to obtain the event horizon at the same (r_+). This interprets the relativistic effect of rotation on the mass distribution in space-time. The minimum value of the $\Delta(r)$ function indicates the potential position of the

event horizon formation. For $a = 0,1$, the minimum value is close to zero at $-0,99$, which indicates that the event horizon is very close to the value of r . As the value of a increases, this minimum value also increases—for example to $-0,75$ for $a = 0,5$, and exactly zero for $a = 1$ which means the horizon shifts to a larger value of r . The maximum value of $\Delta(r)$ also increases as a increases, from $3,01$ at $a = 0,1$ to $4,00$ at $a = 1$, indicating that rotation expands the gravitational horizon. The average $\Delta(r)$ reflects the distribution of values in the range r ; as a gets larger, the average increases from $0,025$ to $1,015$. This shows that rotation strengthens the overall geometric structure of space-time. Meanwhile, the standard deviation $\Delta(r)$ remains constant at $1,116$ for all values of a , which indicates that the shape of the distribution with respect to r remains uniform even though its position is shifted due to the increase in rotation.

Physically, the horizon structure in the *Kerr metric* is significantly affected by the spin parameter a , as shown in Fig. 2. The graph shows the variation of the $\Delta(r)$ function on a logarithmic scale, with the horizontal axis representing the radial distance (r), while the

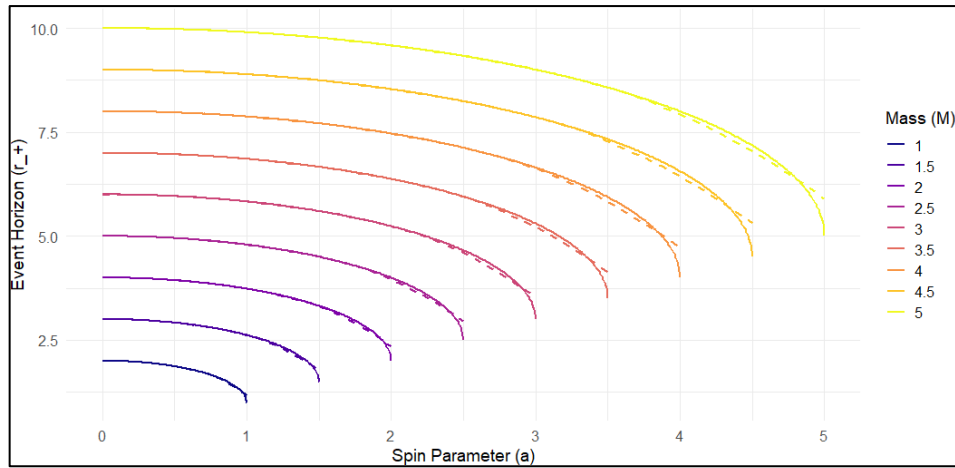


Figure 3. Relationship between spin parameter and outer event horizon in *Kerr metric* for different values of mass (M)

vertical axis depicts the axial coordinate (z). The color of the surface indicates the value of $\log(|\Delta|)$, where the color difference indicates the intensity of the influence of spin on the geometry of space-time [46]. When the value of a is small, the horizon structure appears more symmetric and concentrated around the center, resembling a Schwarzschild solution that has no rotation. However, as the value of a increases, the shape and distribution of Δ become more stretched and distorted, suggesting that rotation causes deformation in the gravitational field around the black hole.

At the extreme value of $a = 1$, which represents the maximum rotation in the Kerr limit, the outer horizon is pushed to a larger radius, while the inner horizon approaches the singularity [47]. This phenomenon is consistent with the prediction of Kerr theory, which states that high rotation drastically changes the horizon structure compared to a non-rotating black hole [48]. In regions farther from the center (for large r and z), the value of $\log(|\Delta|)$ increases, indicating that the effect of rotation becomes weaker as the distance from the singularity increases.

Based on numerical analysis, it is found that the minimum value of r_+ is always equal to the mass (M), which occurs when $a = 0$, corresponding to the Schwarzschild case where the horizon is at $a = 0$. Conversely, the maximum value of r_+ also occurs when $a = 0$, because without rotation, the outer horizon has the maximum distance from the center of the singularity. The average value of r_+ increases with mass, indicating that more massive black holes have larger horizons. There is an increase in the standard deviation, which means that the variation of the horizon due to spin changes is more significant in massive black holes. Thus, in small black holes, spin changes do not affect r_+ much, but in large black holes, rotation has a large impact on the geometry of space-time.

Figure 3 shows the relationship between the spin parameter (a) and the outer event horizon (r_+) in the *Kerr metric* for various values of mass (M). The horizontal axis interprets the value of a , which is also

how fast the black hole rotates, while the vertical axis shows the outer event horizon (r_+), which is the limit where no object, not even light, can escape the gravitational pull of the black hole. The curve on the graph shows that for any value of M , the value of r_+ decreases monotonically as a increases. This is in line with the mathematical solution of the *Kerr metric*, which is $(r_+ = M + \sqrt{M^2 - a^2})$. When $a = 0$, the outer horizon is at the maximum value ($r_+ = 2M$), which corresponds to the Schwarzschild case, i.e. a black hole without rotation. However, as a approaches M , the outer horizon shrinks to the ($a = M$) extremal limit, where the outer and inner horizons meet, forming an extremal Kerr black hole with only one horizon [49]. The different colors on the graph illustrate how (r_+) changes depending on the mass (M); the larger the mass, the higher the curve position, indicating a larger outer horizon [50]. The differences between the curves become more pronounced at large values of M , suggesting that the influence of spin on the horizon is more significant in small-mass black holes [51]. The dashed line resulting from the loess smoothing method shows the general trend of the change in r_+ , illustrating that although this relationship is nonlinear, its behavior can still be predicted consistently in the *Kerr metric* domain.

Analysis of Quadratic Equations

Based on the existing structure of the *Ghasemi-Nodehi-Bambi metric*, it can be seen that b_1 and b_2 play a role in how M appears in the metric components. To maintain compatibility with the *Kerr metric* or its variations, one can equate:

$$\frac{2b_1Mr}{r^2 + b_2a^2\cos^2\theta} = \frac{2Mr}{r^2 + a^2\cos^2\theta} \quad (13)$$

From this, the denominator and numerator can be equalized, which can be interpreted that $b_1 = 1$ and $b_2 = 1$. The metric part corresponding to the coordinates r also provides additional clues:

$$\frac{r^2 + b_5 a^2 \cos^2 \theta}{r^2 - 2b_6 M r + b_7 a^2} \quad (14)$$

In comparing the metric used with the standard forms of the *Kerr-Newman* or *Ghasemi-Nodehi-Bambi metrics*, it can be concluded that the value of b_6 should be equal to 1 in order to remain consistent with the modified Schwarzschild factor. If the value of b_5 is set equal to 1, then the general form that appears in the numerator will be $(r^2 + a^2 \cos^2 \theta)$, which confirms the final form of the mass parameter (M). The resulting quadratic equation of this metric form is used to calculate the values of r and a , taking into account small deviations from the original Kerr solution. For equation M:

$$M = \frac{r^2 + a^2 \cos^2 \theta}{2r} \quad (15)$$

Function (13) depends on the combination of parameters r , a , and θ through the relationship. The standard form equation for r :

$$\begin{aligned} 2b_1 M r &= r^2 + b_2 a^2 \cos^2 \theta \\ r^2 - 2b_1 M r + b_2 a^2 \cos^2 \theta &= 0 \end{aligned} \quad (16)$$

A quadratic equation (r), can be solved using the quadratic formula:

$$r = b_1 M \pm \sqrt{b_1^2 M^2 - b_2 a^2 \cos^2 \theta} \quad (17)$$

This is the solution to r that can be used to calculate the value of r . For equation a is as follows:

$$a = \sqrt{\frac{2b_1 M r - r^2}{b_2 \cos^2 \theta}} \quad (18)$$

As for $\cos \theta$ is:

$$\cos \theta = \pm \sqrt{\frac{2b_1 M r - r^2}{b_2 a^2}} \quad (19)$$

Equation (7), which is represented from the *Ghasemi-Nodehi-Bambi metric*, is expressed in this study by varying the value of M for various values of the parameter a , namely 0.1, 0.5, and 1. The values are obtained by performing calculations on a grid of r values that range from 0.01 to 1, and $\cos \theta$ which is calculated based on the angle (θ) which varies from 0 to 360 degrees.

Relationship between parameter variations metric and thermodynamic properties of black holes

The event horizon is defined as the location where the g_{rr} component of the metric diverges, when g_{rr} becomes zero. From metric:

$$g_{rr} = \frac{r^2 + b_5 a^2 \cos^2 \theta}{r^2 - 2b_6 M r + b_7 a^2} \quad (20)$$

The event horizon occurs when:

$$r^2 - 2b_6 M r + b_7 a^2 = 0. \quad (21)$$

This is a quadratic equation in r . From the above quadratic equation, the coefficients can be identified, so that:

$$r = b_6 M \pm \sqrt{(b_6^2 - b_7)M^2 - b_7 a^2}. \quad (22)$$

In black hole physics in the G-N-B metric, there are two solutions for the horizon, the outer horizon (event horizon) and the inner horizon (Cauchy horizon), which are given by $r_+ = b_6 M + \sqrt{(b_6^2 - b_7)M^2 - b_7 a^2}$ and $r_- = b_6 M - \sqrt{(b_6^2 - b_7)M^2 - b_7 a^2}$ respectively. For both of these solutions to be real-valued, the value inside the square root must be non-negative, i.e. $(b_6^2 - b_7)M^2 - b_7 a^2 \geq 0$. If the condition $(b_6^2 - b_7)M^2 > b_7 a^2$ is satisfied, the square root value is positive, so two different r values are obtained. This implies the existence of two horizons, the outer horizon known as the event horizon, which is the outer limit where neither light nor matter can escape the gravitational pull of the black hole, and the inner horizon or Cauchy horizon, which is the inner limit of the influence of spacetime [52], [53].

However, if the condition changes to $(b_6^2 - b_7)M^2 = b_7 a^2$, then the root becomes zero, which causes the two horizons to meet at a point at $r_+ = r_- = b_6 M$. This state is known as an extremal black hole, where the black hole is in a critical state with horizons that overlap or degenerate into each other [54], [55].

Conversely, if $(b_6^2 - b_7)M^2 < b_7 a^2$, then the value in the root becomes negative, so there is no real solution to the horizon equation. In this condition, the horizon is not formed and the singularity becomes visible or referred to as naked singularity. This phenomenon contradicts the cosmic censorship hypothesis, which states that the singularity should always be hidden behind the horizon and cannot be observed by an outside observer [56].

In the physics of general relativity, the geodesic equation describes the trajectories of objects in space-time that are curved due to the presence of mass and energy. The equation is derived from the space-time metric and is given by:

$$g_{\mu\nu} u^\mu u^\nu = -\epsilon, \quad (23)$$

As for $u^\mu = dx^\mu/d\tau$ as the four-dimensional velocity vector. The value of ϵ determines the type of particle, which is $\epsilon = 1$ for mass particles and $\epsilon = 0$ for photons. To study the dynamics of motion in Kerr geometry, the generalized form of the metric is used:

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2, \quad (24)$$

Since the metric (22) has both time and rotational symmetry, there are two conservative quantities, namely the specific energy (E) associated with time symmetry, and the specific angular momentum (L) associated with rotational symmetry. This relationship is expressed in:

$$p_t = g_{tt}u^t + g_{t\phi}u^\phi = -E, \quad (25)$$

$$p_\phi = g_{\phi\phi}u^\phi + g_{t\phi}u^t = L \quad (26)$$

From here, the components u^t and u^ϕ can be expressed as functions of E and L , which are then substituted into the radial component of the geodesic equation. The motion is considered to be in the equatorial plane, i.e. $\theta = \pi/2$, so the component $u^\theta = 0$, and the radial equation becomes:

$$g_{rr}\left(\frac{dr}{d\tau}\right)^2 = -g_{tt}(u^t)^2 - 2g_{t\phi}u^t u^\phi - g_{\phi\phi}(u^\phi)^2 - \epsilon \quad (27)$$

Substituting the values of u^t and u^ϕ will transform this form into an equation similar to the classical law of conservation of energy:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}} = \text{constant} \quad (28)$$

with an effective potential (V_{eff}) that includes the gravitational and rotational effects of the black hole, namely:

$$V_{\text{eff}} = \frac{1}{2}(g_{tt}E^2 + 2g_{t\phi}EL + g_{\phi\phi}L^2) \quad (29)$$

This potential determines the type of orbit that a particle or light may travel in Kerr space-time. So that from the G-N-B metric in Eqs. 7:

$$g_{tt} = -\left(1 - \frac{2b_1Mr}{r^2 + b_2a^2 \cos^2 \theta}\right) \quad (30.a)$$

$$g_{t\phi} = -\frac{4b_3Mar \sin^2 \theta}{r^2 + b_4a^2 \cos^2 \theta} \quad (30.b)$$

$$g_{\phi\phi} = r^2 + b_9a^2 + \frac{2b_{10}Ma^2r \sin^2 \theta}{r^2 + b_{11}a^2 \cos^2 \theta} \quad (30.c)$$

From the above, the substitution process can be done into the formula V_{eff} :

$$V_{\text{eff}} = \frac{1}{2}\left(1 - \frac{2b_1Mr}{r^2 + b_2a^2 \cos^2 \theta}\right)E^2 - \frac{4b_3Mar \sin^2 \theta}{r^2 + b_4a^2 \cos^2 \theta}EL + \frac{L^2}{2}\left(\frac{1}{r^2 + b_9a^2 + \frac{2b_{10}Ma^2r \sin^2 \theta}{r^2 + b_{11}a^2 \cos^2 \theta}}\right) \quad (31)$$

Changes in these parameter values have direct implications for particle dynamics, particularly in effective gravitational potential and orbital stability.

In black hole physics, the parameter b_1 directly controls the gravitational mass contribution (M) in the metric equation. As the value of b_1 increases, the gravitational pull of the black hole becomes stronger, causing the gravitational potential to deepen. This results in the need for higher escape velocities for particles that want to escape the gravitational field. Conversely, if b_1 is smaller, the gravitational field is weaker, allowing particles to move away more easily. The parameter b_3 relates to the interaction between angular momentum (L) and energy (E) in the effective potential. An increase in b_3 strengthens the link between energy and angular momentum, which affects the circular motion of particles around the black hole. Relativistic effects such as orbit precession become more prominent, where particle trajectories are no longer perfectly elliptical due to space-time distortions [57]. If the value of b_3 is large enough, orbital stability can be compromised and particles are more prone to spiraling towards the event horizon.

The b_4 parameter is related to the frame-dragging effect, a phenomenon in which the space-time around the black hole rotates due to its spin [58]. As b_4 increases, this effect becomes more dominant so that particles near the event horizon are pushed away by the black hole's rotation. In extreme cases, this can expand the region of the ergosphere, allowing particles to gain additional energy through the mechanism of the Penrose Process [59], [60]. Conversely, a smaller value of b_4 will weaken this effect, so that the influence of the black hole's rotation on the motion of the particles becomes smaller.

The three parameters- b_1 , b_3 , and b_4 -significantly modify the curvature of space-time as well as the structure of the event horizon. The b_9 parameter plays a role in determining the effective radius of the event horizon, which means it affects the size of the region that particles cannot travel through without falling into a black hole [61]. Meanwhile, the b_{10} parameter governs the interaction between the gravitational field and angular momentum; its increase can cause orbits to become more unstable and increase the chances of particles falling into the black hole. The b_{11} parameter changes the distribution of curvature around the black hole, which impacts the mechanism of gravitational radiation as well as the gravitational lensing experienced by light and massive particles [62].

The black hole temperature is defined using the surface gravity (κ) [63]:

$$T_H = \frac{\kappa}{2\pi} \quad (32)$$

where the surface gravity (κ) is given by:

$$\kappa = \frac{1}{2}\left|\frac{dg_{tt}}{dr}\right|_{r=r_+} \quad (33)$$

The metric function used for the time component (g_{tt}) is:

$$g_{tt} = 1 - \frac{2b_1Mr}{r^2 + b_2a^2 \cos^2 \theta} \quad (34)$$

It is possible to differentiate g_{tt} with respect to r :

$$\frac{dg_{tt}}{dr} = -\frac{d}{dr} \left(\frac{2b_1Mr}{r^2 + b_2a^2 \cos^2 \theta} \right) \quad (35)$$

The derivative rule can be used, then $f(r) = 2b_1Mr$, so $f'(r) = 2b_1M$, and $g(r) = r^2 + b_2a^2 \cos^2 \theta$, so $g'(r) = 2r$. Substitution into the derivative rule:

$$\frac{dg_{tt}}{dr} = \frac{2b_1M(r^2 - b_2a^2 \cos^2 \theta)}{(r^2 + b_2a^2 \cos^2 \theta)^2}$$

It is possible to take values at the outer horizon (r_+), so that:

$$\kappa = \frac{|b_1M(r_+^2 - b_2a^2 \cos^2 \theta)|}{(r_+^2 + b_2a^2 \cos^2 \theta)^2} \quad (37)$$

Hawking temperature is defined as:

$$T_H = \frac{1}{4\pi} \left| \frac{b_1M(r_+^2 - b_2a^2 \cos^2 \theta)}{(r_+^2 + b_2a^2 \cos^2 \theta)^2} \right| \quad (38)$$

In the relationship between G-N-B metric parameters and black hole temperature, the role of parameters b_1 and b_2 is very important. The b_1 parameter is directly related to the gravitational force generated by the black hole mass. As the value of b_1 increases, the surface gravitational force also increases, which in turn leads to an increase in the black hole temperature [64]. That is, the larger the value of b_1 , the stronger the gravitational field around the event horizon, which affects the emission of Hawking radiation and increases the temperature of the system. In contrast, the b_2 parameter has more to do with the rotational effects of the black hole, specifically through the contribution of the $a^2 \cos^2 \theta$ expression. If the value of b_2 increases, then the expression $r_+^2 - b_2a^2 \cos^2 \theta$ tends to decrease, which translates to a reduced surface gravitational force at the event horizon. Consequently, the Hawking temperature will also decrease. So while both b_1 and b_2 affect the temperature of the black hole, their effects are opposite: an increase in b_1 tends to increase the temperature, while an increase in b_2 can actually decrease it. The entropy of a black hole is given by the Bekenstein-Hawking law of thermodynamics [65], [66]:

$$S = \frac{A}{4}, \quad (39)$$

where A is the area of the event horizon. For the G-N-B metric, the horizon area is calculated by:

$$A = \int_0^{2\pi} \int_0^\pi \sqrt{g_{\theta\theta}g_{\phi\phi}} d\theta d\phi \quad (40)$$

From the G-N-B metric, the two-component metrics for horizon area calculation are, Component $g_{\theta\theta}$:

$$g_{\theta\theta} = r_+^2 + b_8a^2 \cos^2 \theta \quad (41)$$

The component $g_{\phi\phi}$:

$$g_{\phi\phi} = r_+^2 + b_9a^2 + \frac{2b_{10}Ma^2r_+ \sin^2 \theta}{r_+^2 + b_{11}a^2 \cos^2 \theta} \quad (42)$$

The horizon area is obtained from the integral:

$$A = \int_0^{2\pi} \int_0^\pi \sqrt{g_{\theta\theta}g_{\phi\phi}} d\theta d\phi \quad (43)$$

Substitute the values of $g_{\theta\theta}$ and $g_{\phi\phi}$:

$$A = \int_0^{2\pi} \int_0^\pi \sqrt{(r_+^2 + b_8a^2 \cos^2 \theta) \left(r_+^2 + b_9a^2 + \frac{2b_{10}Ma^2r_+ \sin^2 \theta}{r_+^2 + b_{11}a^2 \cos^2 \theta} \right)} \sin^2 \theta d\theta d\phi \quad (36)$$

The above calculation can be done with a memorization, where

$$X = r_+^2 + b_8a^2 \cos^2 \theta, \quad (45)$$

$$Y = r_+^2 + b_9a^2 + \frac{2b_{10}Ma^2r_+ \sin^2 \theta}{r_+^2 + b_{11}a^2 \cos^2 \theta} \quad (46)$$

So from this, It can be seen from equation (47) that:

$$A = \int_0^{2\pi} d\phi \int_0^\pi \sqrt{XY} \sin \theta d\theta \quad (47)$$

The integral over ϕ becomes a factor of 2π , so that:

$$A = 2\pi \int_0^\pi \sqrt{XY} \sin \theta d\theta \quad (48)$$

For simplicity, a Taylor approximation can be used on b_8, b_9, b_{10}, b_{11} assuming b_i is small:

$$X \approx r_+^2 + b_8a^2 \cos^2 \theta, \quad (49)$$

$$Y \approx r_+^2 + b_9a^2 + \frac{2b_{10}Ma^2r_+ \sin^2 \theta}{r_+^2 + b_{11}a^2} \quad (50)$$

From this, the dominant tribe can be taken:

$$A \approx 2\pi \left[r_+^2 + \frac{b_8a^2}{2} + \frac{b_9a^2}{2} \right] \quad (51)$$

$$A \approx 4\pi \left[r_+^2 + \frac{b_8a^2}{2} + \frac{b_9a^2}{2} + \frac{b_{10}Ma^2r_+}{r_+^2 + b_{11}a^2} \right] \int_0^\pi \sin \theta d\theta \quad (52)$$

As known for the entropy equation (S), one can substitute the value of A :

$$S \approx \pi \left[r_+^2 + \frac{b_8a^2}{2} + \frac{b_9a^2}{2} + \frac{b_{10}Ma^2r_+}{r_+^2 + b_{11}a^2} \right] \quad (53)$$

If b_8 increases, the horizon area increases, so entropy increases. If b_9 increases, entropy also increases. If b_{10} increases, then the contribution to the horizon area depends on the ratio $\{Ma^2r_+/(r_+^2 + b_{11}a^2)\}$, which can increase or decrease entropy. If b_{11} increases, then the additional term in the horizon area decreases, so the entropy decreases.

Impact of Factor Variations on M-Value Distribution and System Sensitivity

In general relativity physics, the variation of the spin factor a has a significant impact on the distribution of the M value, which is the sensitivity of the system to the rotation of the black hole. Based on the research conducted, it can be seen that when the value of a increases from 0.1 to 1, the range of M values also expands strikingly. At the value of $a = 0.1$, the value of M is still confined within a narrow range, with a minimum value of 0.0055 and a maximum of 0.505. The

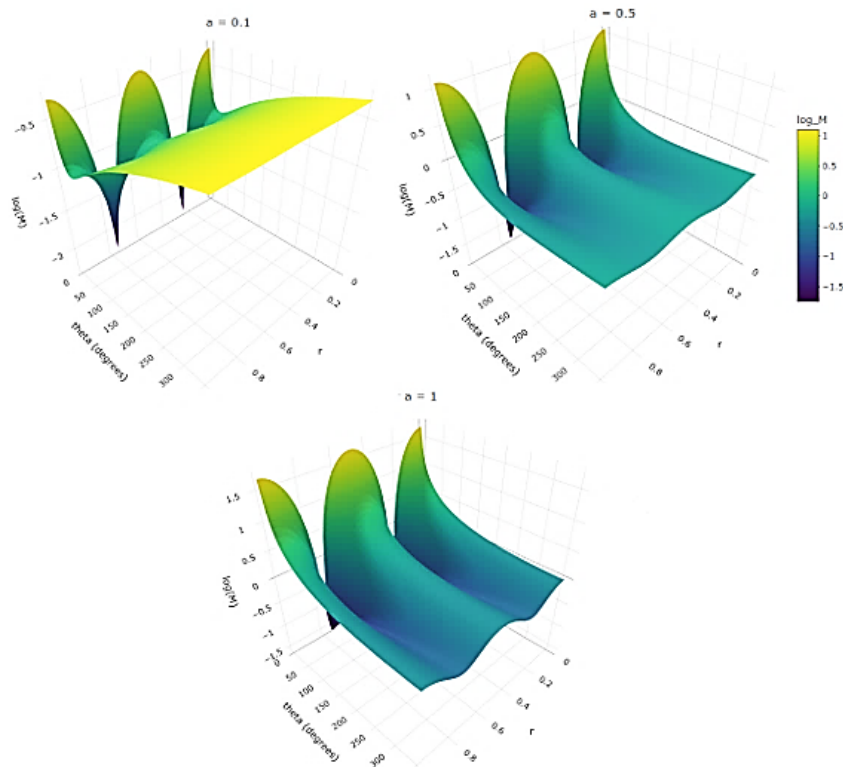


Figure 4. Effect of Parameter Spin on the *Ghasemi-Nodehi-Bambi metric*

mean and median are both around 0.267, which indicates that the influence of rotation on the system is still relatively small and the values of M tend to be concentrated in the low range.

However, when the value of (a) increases to 0.5, there is a drastic change in the distribution of M values. The range extends from 0.0178 to 12.505, with the mean rising to 0.624 and the median to 0.466. This shows that the contribution of the rotation factor to the metric is getting larger, increasing the variability in the geometric calculation of the system. At the extreme condition, i.e. $a = 1$, the distribution of M values gets wider and less symmetrical, with a minimum value of 0.0321 and a maximum reaching 50.005. The mean increases dramatically to 1.741, and the median reaches 0.851. This large difference between the median and maximum indicates that most of the values are still distributed in the low range, but there are sharp deviations due to the high rotation contribution.

An increase in the value of a means an increase in the sensitivity of the system to rotational effects. The greater variability of the M value means that the *Kerr metric* system becomes more responsive to rotational changes, and this illustrates the dominance of the a factor in influencing the space-time structure around a rotating black hole.

Impact of Parameter a by 3D Logarithmic Visualization

The three-dimensional visualization with logarithmic scale in Figure 4 shows that the parameter a has a

significant influence on the *Ghasemi-Nodehi-Bambi metric*. When $a = 0.1$, the changes in the value of the metric M are relatively small and concentrated at low values, which means that the system is in a stable state and responsive to controllable changes. As the value of a increases to 0.5, the variation of M value also increases, indicating that the system becomes more sensitive to changes in r and θ coordinates. When $a = 1$, the fluctuation of M value becomes very large, indicating that the influence of a higher a parameter causes much more extreme system changes. Hence, the greater the value of a , the greater the change in the metric, indicating that the system is more responsive to variations in the parameter.

Sensitivity Analysis of Metric Behavior

A sensitivity analysis of the metric behavior shows that the values of the parameter a calculated based on the variation of the black hole mass (M), radial distance (r), and polar angle (θ) result in a very large range, from zero to more than 1.63×10^{16} . As for most of the a values, they are distributed very close to zero. This phenomenon can be explained by the role of $\cos^2(\theta)$ in the denominator in the metric function, which causes the value of a to become very small when θ approaches 0° or 180° , and becomes much larger when θ approaches 90° . The average value of a is recorded as 3.11×10^{14} , which indicates a significant difference in distribution depending on the parameter variation. The maximum value of a reaching 1.63×10^{16} is much higher than both the mean and median values, indicating that certain

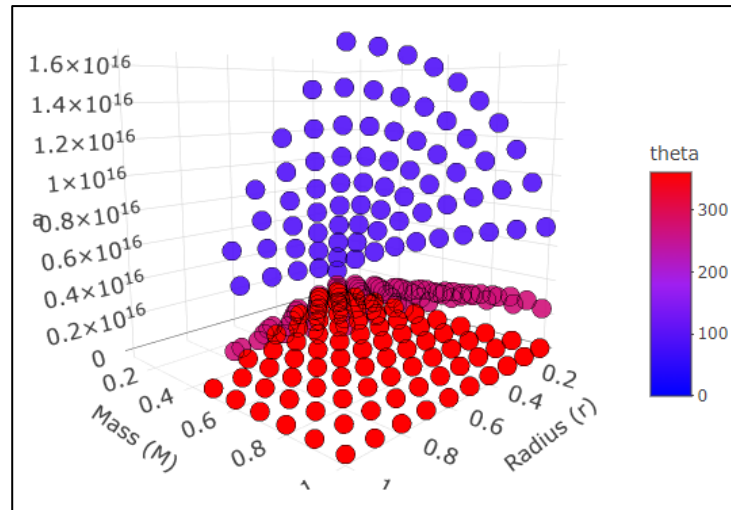


Figure 5. Relationship between mass (M), radius (r), and value spin parameters, with angle which is depicted with color

parameters, especially the θ angle, will contribute greatly to the results.

As for this distribution, it shows that the metric system is very sensitive to polar and radial angle variations, especially when spin a approaches high values. In the context of the *Ghasemi-Nodehi-Bambi metric*, low values of spin $a = 0.1$ lead to small and stable changes in the metric M , interpreting a responsive yet controllable system. However, as the spin value increases to 0.5 or even 1, the metric variations become increasingly large, and the sensitivity of the system to changes in r and θ increases sharply, causing extreme fluctuations in the resulting space-time structure.

Non-Linear Variations

In the *Ghasemi-Nodehi-Bambi metric*, the value of a is strongly influenced by the relationship between mass (M), radius (r), and angle (θ). When θ approaches extreme values, such as 0° or 180° , it has a great impact on the value of (a), causing extreme fluctuations. This phenomenon means that the non-linear behavior is very sensitive to small changes in the parameters entered into the equation.

The study found that the distribution of a values calculated from various combinations of M , r , and θ , with a very wide range of a values, ranging from 0 to exceeding 1.63×10^{16} . Most a values are concentrated around zero, affected by the $\cos^2(\theta)$ term, with higher values occurring when θ approaches 90° or 270° . The average value of a is about 3.11×10^{14} , suggesting there is significant variation in a based on these parameters.

Non-Linear Response of (Theta) to Mass, Radius, and Angle Variations

Figure 5 shows the relationship between the parameters mass (M), radius (r), and value (a) with angle (θ) represented by color. Each point on the graph represents a specific combination of M , r , and θ , with the size of the

point indicating the value of a . Physically, this illustrates how a changes as mass, radius, and angle (θ) vary. When θ approaches extreme values such as 0° or 180° , the value of a becomes very small due to the effect of the term $\cos^2(\theta)$ in the denominator. Conversely, as θ approaches 90° or 270° , a increases. This analysis is based on equation of the *Ghasemi-Nodehi-Bambi metric*.

The value of a shows significant variation based on the combination of M , r , and θ , ranging from 0 to more than 1.63×10^{16} . Most values are clustered around zero due to the influence of the term $\cos^2(\theta)$, which reduces a as θ approaches 0° or 180° , and increases it around 90° or 270° . The average value of a is about 3.11×10^{14} , with a maximum of 1.63×10^{16} , which shows a large variability. These results interpret the strong sensitivity of a to the interactions between M , r , and θ , especially around the extreme values of θ , which interpret the non-linear response describing the fundamental physics of this system.

Comparison of the advantages of Ghasemi-Nodehi-Bambi (G-N-B) metric over Kerr metric in modeling black hole thermodynamics

A comparison between the advantages of the Ghasemi-Nodehi-Bambi (G-N-B) metric and the *Kerr metric* in black hole thermodynamic modeling reveals important differences in the relationship between black hole temperature and entropy. In the analysis of black hole thermodynamics, the Hawking temperature is a fundamental quantity related to the surface gravity of the event horizon. In the *Kerr metric*, the temperature depends directly on the spin parameter a . The larger the value of a , the weaker the surface gravity at the event horizon, which causes the Hawking temperature to decrease. This implies that the rotation effect weakens the gravitational field around the horizon, making it more difficult for the virtual particles responsible for Hawking radiation to escape its pull. Therefore, the larger the value of a , the "colder" the black hole becomes, until at

the extreme $a = M$, the temperature drops to zero, indicating that Hawking radiation is no longer occurring.

However, in the Ghasemi-Nodchi-Bambi (G-N-B) metric, the relationship between Hawking temperature and black hole parameters becomes more complex. Besides depending on the mass (M) and spin (a), the temperature in the model is affected by an additional factor in the metric that changes the structure of surface gravity. This causes the relationship between temperature and black hole parameters to no longer be as simple as in the *Kerr metric*, but instead involves richer non-linear interactions. The presence of additional parameters in the G-N-B metric allows for a more flexible variation of the horizon structure, so small changes in the metric parameters can result in significant differences in the temperature distribution around the event horizon.

For black hole entropy, there is a notable difference between the Kerr and G-N-B metrics. In the *Kerr metric*, the entropy is completely determined by the event horizon area based on the classical relation $S = A/4$, where A is the horizon area. Since in general relativity the horizon area is a function of the mass and spin of the black hole, the entropy change in the *Kerr metric* depends only on these two parameters. In contrast, in the G-N-B metric, entropy depends on the event horizon area, and is also affected by the polar angle (θ). This angular variation means that the entropy distribution in a black hole is not isotropic, but depends on a particular spatial orientation. The G-N-B metric allows for a model where the black hole entropy can vary in different directions, enriching the thermal dynamics that cannot be represented in Kerr geometry. The dependence shows that the influence of rotational effects on the thermodynamic properties of black holes in the G-N-B metric is more complex than in the *Kerr metric*, opening up possibilities for further exploration in the radiation mechanism and thermal evolution of black holes.

Discussion

Research has used quadratic equations to analyze the effect of parameters on the shape of mass (M) in metric components, particularly in the modified *Kerr metric*. As a solution to Einstein's field equations for rotating black holes, the *Kerr metric* initially does not take into account the small deviations that occur due to variations in certain parameters, such as rotation or mass parameters. Therefore, in order to understand the changes in the metric structure due to the influence of these parameters, it is necessary to modify it through mathematical relationships that remain compatible with the original Kerr solution. According to existing theory, small changes in these parameters will affect the mass distribution and the shape of spacetime around the black hole [67], [68]. In this case, the quadratic equation is used to determine the radial coordinate (r) that has a small deviation from the ideal Kerr solution. Previous research, such as that conducted by [69] and [70], shows that the *Kerr metric* solution remains consistent if minor

modifications are made, as long as the right mathematical relationships are applied. Through quadratic equations, it is possible to model how the values of the radian and rotation parameters affect the location of the horizon and other properties of the spacetime around the black hole, which can be adjusted to a more accurate solution according to parameter changes. This finding supports the opinion put forward by previous studies that although the *Kerr metric* provides a basic description of a rotating black hole, small variations in the parameters can change the dynamics of the spacetime, which can only be analyzed by accounting for deviations using quadratic equations to maintain conformity with the original form [71].

Research has shown that variations in metric parameters have a significant influence on the thermodynamic properties of black holes, which is consistent with previous theories on black hole thermodynamics. Within the framework of the general theory of relativity, the structure of the space-time metric describing the black hole, specifically the *Kerr metric*, serves to determine the position of the event horizon, which is created when certain components in the metric reach zero. This process results in a quadratic equation in radial coordinates, which is used to calculate the location of the event horizon. This is in line with the results of previous research, as described by [52] which links the temperature of a black hole to the spacetime geometry around the event horizon. This discovery emphasizes how changes in metric parameters, such as the black hole's mass and angular momentum, affect the black hole's thermodynamics, including its temperature and entropy, which are directly related to the size of the event horizon. As the mass of the black hole increases, the temperature and entropy of the black hole also increase, as discussed by [72] in the context of black holes with rotation.

Research has examined the existence of horizons in the *Kerr metric*, which are affected by the solution of quadratic equations in radial coordinates. Based on previous theory, two horizons are formed, namely the outer horizon or event horizon that serves as the absolute gravitational limit, as well as the inner horizon or Cauchy horizon that limits the region where the solution of the metric remains analytically definable [73], [74]. The existence of horizons depends largely on the discriminant of the quadratic equation associated with the metric component. If the discriminant is positive, two horizons are formed, known as the non-extremal condition, where the two horizons such as the outer and inner horizons are clearly separated. On the other hand, when the discriminant is zero, these two horizons meet at a single point, resulting in the extremal condition, where the outer and inner horizons become one. In this case, the event horizon and the Cauchy horizon are no longer separate. In the negative discriminant condition, no horizon is formed, which leads to a naked singularity, a state that contradicts the principle of the Cosmic Censorship Hypothesis, which states that the singularity

must always be hidden inside the horizon [33]. This finding is in line with previous studies showing that the existence of horizons in the *Kerr metric* solution depends on mathematical relationships that are very sensitive to the values of the mass (M) and angular momentum parameters, which affect the space-time structure around the black hole [71], [75].

In general relativity, the frame-dragging effect is one of the most distinctive phenomena arising from black hole rotation, as described in Kerr's solution to Einstein's field equations [76], [77]. Research has shown that variations in rotation parameters significantly affect the strength of the frame-dragging effect. As the rotation parameter increases, the space-time around the black hole rotates more strongly, leading to more intense frame-dragging [58]. This condition creates a zone known as the ergosphere, which is the region beyond the event horizon where no particle can be stationary with respect to a distant inertial frame [78]. Within this zone, particles will be trapped, and potentially gain additional energy through a mechanism known as the Penrose process, where partially fragmented particles can fall into the black hole while providing energy to their escaping fragments [79]. This finding is consistent with the theory proposed by Roger Penrose in 1969, which states that energy can be extracted from rotating black holes through particle interactions within the ergosphere [80]. As for previous numerical and analytical research such as by [81], [82] has shown that an increase in the angular momentum of black holes leads to an expansion of the ergosphere and an increase in the potential for energy extraction.

The study found that the temperature and entropy of black holes show a complex relationship to the mass (M) and rotation a parameters, in line with the thermodynamic theory of black holes in general relativity. The black hole temperature, defined by the surface gravity at the event horizon, tends to increase as the mass (M) increases, as the resulting gravitational field becomes stronger [83]. However, increasing the rotation parameter a decreases the temperature. This is consistent with the *Kerr metric* solution, where the event horizon expands due to rotation, which in turn decreases the local curvature of spacetime and leads to reduced surface gravity [84], [85]. This finding is consistent with the formulation [86], where the black hole temperature is inversely proportional to the horizon radius and depends on the metric structure. Meanwhile, black hole entropy, according to the Bekenstein-Hawking principle, depends directly on the event horizon area [87]. Therefore, an increase in mass (M) that expands the horizon will directly increase entropy. However, if the rotation of a is large enough, the horizon may narrow, thus decreasing its area and causing the entropy to decrease. This phenomenon has been discussed in previous studies by [88] and [89], which emphasizes that in extreme rotating systems, the horizon can decrease significantly enough to substantially affect the entropy value.

In the G-N-B metric framework, the dynamics of particle motion is analyzed by considering the laws of energy conservation and angular momentum, two fundamental principles in general relativistic mechanics. The mass parameter (M) plays a direct role in determining the intensity of the gravitational field generated by the black hole. In line with the general theory of relativity proposed by Einstein, an increase in the value of M enlarges the curvature of the surrounding space-time, so the escape speed of particles from the field also increases. This is consistent with the classical studies by Misner, Thorne, and Wheeler on the [90], which states that larger gravitational fields require higher kinetic energy to escape their influence. Meanwhile, the a parameter, which describes the black hole's angular momentum or its rotational speed, introduces an additional relativistic effect known as frame-dragging. In this phenomenon, space-time itself is rotated by the rotating massive object, causing surrounding particles to experience a pull in the direction of rotation. This effect was first described in the Kerr solution and has been proven through experimental observations. In the G-N-B metric, larger values of a amplify this effect, causing significant changes in the trajectories of particles depending on their direction of motion with respect to the rotation of the black hole. Particles moving in the direction of rotation will experience acceleration, while those moving in the opposite direction will experience deceleration. This finding supports previous theoretical predictions and extends the understanding of relativistic dynamics in modified spacetime geometries.

The results show that the distribution of the black hole mass value (M) is very sensitive to the variation of the rotation parameter a , which intercepts the black hole angular momentum in the *Kerr metric* framework. When the value of a is at a low level, the variation of M value tends to be small and the system shows a relatively constant metric stability. However, as the value of a increases to intermediate levels, the sensitivity of the system to parameter variations becomes more pronounced, characterized by the appearance of significant fluctuations in the M distribution, due to the nonlinear interaction between r (radial coordinates) and a itself. At high values of a , the system shows a very sensitive response to parameter changes, so that the distribution of M values becomes much wider and is characterized by extreme fluctuations. This indicates that the metric dynamics in the modified Kerr system gets more complex at high rotations, and the space-time stability is disturbed by intense frame-dragging effects. This finding is consistent with the general theory of relativity, in which the increase in black hole angular momentum enlarges the ergospheric zone and significantly affects the structure of the event horizon [91], [92]. Furthermore, the increase in the mean and median values of M as a increases indicates that the rotation parameter has a dominant influence on the distribution of metric values. This reinforces the idea that the system is dynamical rather than static, with a metric structure that strongly depends on the space-time

geometry determined by the a parameter. This result is also in line with the thermodynamic approach of black holes, where changes in a directly affect the horizon properties, surface temperature and entropy of the black hole [87], [93], thus confirming that rotational variations affect the geometric structure and the entire physical dynamics of the system.

CONCLUSION

The theoretical conclusions of the study show that the quadratic equation plays a role in determining the effect of the M parameter on the modified metric structure of Kerr, with solutions in radial coordinates (r) affecting the spacetime dynamics. The event horizon depends on the discriminant of the quadratic equation, which can result in two horizons in the non-extremal state, one horizon in the extremal state, or even no horizon in the case of a bare singularity. The frame-dragging effect gets stronger as the rotation spin parameters increases, creating an ergosphere that allows energy extraction through the Penrose process. The black hole temperature increases with increasing M but decreases with increasing of spin, while the entropy is directly proportional to the event horizon area. In the G-N-B metric, the conservation of energy and angular momentum determines the particle trajectory, where M amplifies the gravitational field and increases escape velocity, while spin causes trajectory changes due to frame-dragging effects. The variation of the angle plays a role in the distribution of space-time curvature, which affects the stability of particle orbits.

Parameter variations in the G-N-B metric allow a deeper analysis of black hole characteristics, including thermodynamic properties and the structure of the event horizon in various physical configurations. Understanding the horizon and extremal conditions contributes to the identification of black hole properties in astrophysical observations. Frame-dragging effects have important implications in the detection of gravitational waves as well as particle dynamics around rapidly rotating massive objects. The study of black hole thermodynamics is central to the study of their evolution and stability, especially in relation to radiation and energy transfer. The sensitivity of the M parameter to spin shows that rotation plays a key role in the gravity distribution, which can be applied in simulations of black holes and strong gravity interactions.

Future research can explore the effect of additional parameter variations, such as electric charge or external field, on the dynamics of black holes in the G-N-B metric. Further numerical analysis can be done to understand the stability of particle orbits under extreme conditions. The study of quantum effects and Hawking radiation in the system delves deeper into the thermodynamic evolution of black holes. The development of observational data-based simulation models is also recommended to compare theoretical predictions with astrophysical observations.

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